Problem sheet 9

Tutorials by Mohammad Hashemi <hashemi@math.uni-leipzig.de>.
Solutions will be collected during the lecture on Tuesday January 7.

1. [**1+2 points**] Let $f(z) = z^2$, $z \in \mathbb{C}$.
   (a) Determine the angle of rotation of the complex plane by f at the point $z = 1 + i$.
   (b) Which part of the complex plane is stretched and which is contacted by f?

2. [**3 points**] Find the image of the interior of the circle $\gamma : \ |z - 2| = 2$ under the linear fractional transformation $w = f(z) = \frac{z}{2z-8}$. Sketch the image and pre-image of γ under $w = f(z)$.

3. [**3 points**] Show that the linear fractional transformation $f(z) = \frac{a(z-z_0)}{\bar{z}_0 z - 1}$ maps the disc $B = \{z \in \mathbb{C} : \ |z| < 1\}$ onto itself, where $|z_0| < 1$ and $|a| = 1$ are some complex numbers.

4. [**3 points**] Let γ be a continuously differentiable positively oriented boundary of a set $S \subset \mathbb{C}$ with area A. Compute the integral $\int_\gamma \operatorname{Re} z \, dz$.

5. [**1+2+3 points**] Evaluate the complex line integral $\int_\gamma f(z)dz$ in the following cases.
   (a) $f(z) = z^3$, γ is a part of the parabola $x = y^2$, that connects the points 0 and $1 + i$ in the complex plane.
   (b) $f(z) = |z|$, γ is the half circular $|z| = 1$, $0 \le \arg z \le \pi$ ($z = 1$ is the initial point);
   (c) $f(z) = |z|\bar{z}$, γ is the union of the half circular $|z| = 1$, $y \ge 0$, and the segment $-1 \le x \le 1$, $y = 0$,

6. [**5 points**] Prove that $\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$.

   (*Hint:* Integrate the function $f(z) = e^{iz^2}$ along the boundary of the domain $0 \le |z| \le R$, $0 \le \arg z \le \frac{\pi}{4}$, and then pass to the limit as $R \to \infty$.)