Problem sheet 1

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Solutions will be collected during the lecture on Monday October 28.

1. [**2+2 points**] a) Show that a set of Lebesgue measure zero has no interior points.

   b) Construct a set having Lebesgue measure zero whose closure is the entire space $\mathbb{R}^d$.

2. [**3 points**] Show that a union of a finite or countable number of sets of Lebesgue measure zero is a set of Lebesgue measure zero.

3. [**4 points**] Let I be a rectangle in $\mathbb{R}^d$ and $f : I \to \mathbb{R}$ be bounded. Let, for a partition P of I, $L(f, P)$ and $U(f, P)$ denote the lower and upper Darboux sums, respectively. Using the Darboux criterion, show that f is integrable over I if and only if for every $\varepsilon > 0$ there exists a partition P of I such that $U(f, P) - L(f, P) < \varepsilon$.

4. [**4 points**] Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Show that the graph of f

   $$\mathrm{Gr} = \{(x, f(x)) : \ x \in [0, 1]\}$$

   has measure zero in $\mathbb{R}^2$.

5. [**2 points**] Prove that $\partial S = \bar{S} \setminus S^\circ$ for any set $S \subset \mathbb{R}^d$, where $\bar{S}$ and S° denote the closure and the interior of S, respectively.

6. [**2 points**] Let $S_1, S_2 \subset \mathbb{R}^d$. Show that $\partial(S_1 \cap S_2) \subset \partial S_1 \cup \partial S_2$.

7. [**3+1 points**] a) Show that if a set $S \subset \mathbb{R}^d$ is such that $\mu(S) := \int_S dx$ exists and $\mu(S) = 0$, then $\mu(\bar{S})$ also exists and equals zero for the closure $\bar{S}$ of the set S.

   b) Give an example of a bounded set $S \subset \mathbb{R}^d$ of Lebesgue measure zero whose closure $\bar{S}$ is not a set of Lebesgue measure zero.