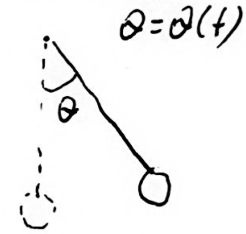
(1)

N26 Systems of linear differential equations.

1. Rewriting scalar differential equations as systems.

Ex 26.1 Consider the equation for the motion of a gravitational pendulum (with length 1)



(26.1) $$\theta'' + g \sin\theta = 0$$

We denote the velocity of the motion by v. So

$$\theta' = v$$

Then (26.1) we can rewrite as

$$\begin{cases} v' = -g \sin\theta \\ \theta' = v \end{cases}$$

– a system of diff. equations.

Ex 26.2 Let us consider a higher order linear equation with constant coefficients:

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = F, \quad a_0 \neq 0$$

Similarly, let

$$y_1 := y$$

$$y_2 := y' = y_1'$$

$$y_3 := y'' = y_2'$$

$$\dots$$

$$y_n := y^{(n-1)} = y_{n-1}'$$

Then

$$\overset{y^{(n)}}{\overset{\parallel}{y_n'}} = -\frac{a_1}{a_0} y_n - \frac{a_2 y_{n-1}}{a_0} - \dots - \frac{a_{n-1}}{a_0} y_2 - \frac{a_n}{a_0} y_1 + F$$

So, we have obtain the following system:

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ \dots \\ y_{n-1}' = y_n \\ y_n' = -\frac{a_1}{a_0} y_n - \dots - \frac{a_n}{a_0} y_1 + F \end{cases}$$

<u>Def 26.1</u> A system of the form

$$\begin{cases} y_1' = g_1(t, y_1, \dots, y_n) \\ \dots \\ y_n' = g_n(t, y_1, \dots y_n) \end{cases}$$

is called a <u>first order system</u> of differential equations.

③ **2. Linear systems of diff. equations**

Def 26.2 A first order system of diff. eq. that can be written in the form

$$(26.2)\quad \begin{cases} y_1' = a_{11}(t)y_1 + a_{12}(t)y_2 + \dots + a_{1n}(t)y_n + f_1(t) \\ y_2' = a_{21}(t)y_1 + a_{22}(t)y_2 + \dots + a_{2n}(t)y_n + f_2(t) \\ - - - - - - - - - - - - \\ y_n' = a_{n1}(t)y_1 + a_{n2}(t)y_2 + \dots + a_{nn}(t)y_n + f_n(t) \end{cases}$$

is called a <u>linear system of diff. eq.</u>

The system (26.2) can be rewritten as

$$y' = A(t)y + f(t),$$

where $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, $A(t) = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ - & - & - \\ a_{n1}(t) & \dots & a_{nn}(t) \end{pmatrix}$

$$f(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}.$$

The matrix A is called the <u>coefficient matrix</u> of (26.2). If $f = 0$, then (26.2) is called <u>homogeneous</u>.

Th 26.1 Let A and f be continuous on (a,b) and let $t_0 \in (a,b)$, $y_0 \in \mathbb{R}^n$. Then the system

$$y' = A(t)y + f(t), \quad y(t_0) = y_0$$

has a unique solution on (a,b).

(4) 3. Homogeneous system of linear equations

In this section we consider a system

(26.3) $\quad y' = A(t) y.$

It is trivial to see that $y = 0$ is a solution to (26.3). It is called a trivial solution.

Let $y^1, \dots, y^n$ (not power!) be a vector valued functions which are solutions to (26.3). Then for any constants $C_1, \dots, C_n$

$$y = C_1 y^1 + \dots + C_n y^n \qquad (26.4)$$

is also solution to (26.3).

Def 26.3 If any solution to (26.3) can be written in the form (26.4) for some constants $C_1, \dots, C_n$, then $y^1, \dots, y^n$ is called a fundamental system of solutions to (26.3).

Def 26.4 $y^1, \dots, y^n$ is called linearly independent on (a,b) if the equality

$$C_1 y_1(t) + \dots + C_n y_n(t) = 0 \quad \forall t \in (a,b)$$

implies $C_1 = \dots = C_n = 0.$

Th 26.2 Let the $n\times n$-matrix $A = A(t)$ be continuous on (a,b). Then a set $y^1, \dots, y^n$ of n solutions to (26.3) on (a,b) is a fundamental system of solutions iff it is linearly independent on (a,b).

4. Solving of homogeneous systems of linear equations with constant coefficients.

We assume that the matrix of (26.3) is constant. Let us find solutions to (26.3) (with constant coefficients) as solutions to higher order linear equations.

Ex.26.3 Consider a system

$$y' = \begin{pmatrix} -4 & -3 \\ 6 & 5 \end{pmatrix} y, \quad \text{where } y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

or

$$\begin{cases} y_1' = -4y_1 - 3y_2 \\ y_2' = 6y_1 + 5y_2 \end{cases} \qquad (26.5)$$

We will find solutions in the form

$$y_1 = x_1 e^{\lambda t}, \quad y_2 = x_2 e^{\lambda t}, \qquad (26.6)$$

where x_1, x_2 some constants.

⑥

Then $y_1' = \lambda x_1 e^{\lambda t}$, $y_2' = \lambda x_2 e^{\lambda t}$ (26.7)

Substituting (26.6), (26.7) into (26.5), we obtain

$$\begin{cases} -4x_1 - 3x_2 = \lambda x_1 \\ 6x_1 + 5x_2 = \lambda x_2 \end{cases}$$

or

$$\begin{cases} (-4-\lambda)x_1 - 3x_2 = 0 \\ 6x_1 + (5-\lambda)x_2 = 0 \end{cases} \qquad (26.8)$$

(It is a system of linear eq. $(A-\lambda I)x = 0$, where $x = \binom{x_1}{x_2}$)

Then we have to find λ for which it has nontrivial solution. We find λ from the equation

$$\begin{vmatrix} -4-\lambda & -3 \\ 6 & 5-\lambda \end{vmatrix} = 0$$

$$\Updownarrow$$

$$(-4-\lambda)(5-\lambda) + 18 = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1) = 0.$$

So, $\lambda_1 = 2$, $\lambda_2 = -1$.

1) Take $\lambda_1 = 2$ and find x_1, x_2 from (26.8).

$$\begin{cases} -6x_1 - 3x_2 = 0 \\ 6x_1 + 3x_2 = 0 \end{cases}$$

Hence $x_1 = -1$, $x_2 = 2$

Hence

$y_1 = -e^{2t}$, $y_2 = 2e^{2t}$

or $y' = \binom{-1}{2} e^{2t}$.

7

2) For $\lambda_2 = -1$ we obtain from (26.8)

$$\begin{cases} -3x_1 - 3x_2 = 0 \\ 6x_1 + 6x_2 = 0 \end{cases}$$

So, $x_1 = -1, \ x_2 = 1$

$y_1 = -e^{-t}, \ y_2 = e^{-t}$ or

$$y^2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

Hence, the general solution to (26.5) is given by

$$y = C_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

or

$$y_1 = -C_1 e^{2t} - C_2 e^{-t}$$
$$y_2 = 2C_1 e^{2t} + C_2 e^{-t}$$

In order to solve

$$y' = Ay \qquad (26.9)$$

first we find eigenvalues of the matrix A from the equation

$$\det(A - \lambda I) = 0.$$

Let $\det(A - \lambda I) = (\lambda - \lambda_1)^{k_1} \dots (\lambda - \lambda_m)^{k_m}$.

8

1) If $\lambda := \lambda_j \in \mathbb{R}$ and the fundamental system of solutions to

$$(A - \lambda I) x = 0 \qquad (26.10)$$

consists of $k := k_j$ solutions $x^1, \dots, x^k$, then

$$y^1 = x^1 e^{\lambda t}, \dots, y^k = x^k e^{\lambda t}$$

are linearly independent solutions to (26.9)

2) If $\lambda \in \mathbb{R}$ and the fundamental system of solutions consists of $m < k$ solutions. Then find solutoons to (26.9) in the form

$$y = \left(x^{(1)} + x^{(2)} t + \dots + x^{(k-m)} t^{k-m}\right) e^{\lambda t}.$$

3) If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then find solutions to (26.9) as before. We obtain

$$y = y^1 + i y^2.$$

Then take y^1, y^2 as linearly independent solutions to (26.9).

Ex 26.4

$$\begin{cases} y_1' = 2y_1 + y_2 + y_3 \\ y_2' = -2y_1 - y_3 \\ y_3' = 2y_1 + y_2 + 2y_3 \end{cases} \qquad (26.11)$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ -2 & -\lambda & -1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$$ – characteristic equation

Solving it, we obtain

$\lambda_1 = 2, \quad \lambda_2 = \lambda_3 = 1.$

1) $\lambda_1 = 2$

$$\begin{cases} x_2 + x_3 = 0 \\ -2x_1 - 2x_2 - x_3 = 0 \\ 2x_1 + x_2 = 0 \end{cases}$$

Find fundamental system of solutions

$$\left(\begin{array}{ccc|ccc} 0 & -2 & 2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \overset{II-III}{\sim} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & -1 \\ 0 & -2 & 2 & 1 & 0 & 0 \end{array}\right) \underset{III-2II}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -2 & 2 \end{array}\right)$$ ← fundam. syst. of solutions

$$\begin{aligned} y_1^1 &= e^{2t} \\ y_2^1 &= -2e^{2t} \\ y_3^1 &= 2e^{2t} \end{aligned} \qquad \text{or} \qquad y^1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} e^{2t}.$$

(10)

2) $\lambda = 1$ $(k=2)$ First we find the number of vectors in fundamental system of solutions to

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ -2x_1 - x_2 - x_3 = 0 \\ 2x_1 + x_2 + x_3 = 0 \end{cases}$$

For this we compute the rank of matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & -1 \\ 2 & 1 & 1 \end{pmatrix} \underset{\begin{smallmatrix} I+II \\ I+III \end{smallmatrix}}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Its rank $= 2$. So, the fundam. syst. of sol. contains only $m = 3-2 = 1$ vector.

Find solutions in the form

$$y_1 = (x_1 + z_1 t)e^t$$
$$y_2 = (x_2 + z_2 t)e^t$$
$$y_3 = (x_4 + z_3 t)e^t$$

Substituting them into the system (26.11) we get

$$\begin{cases} z_1 + z_2 + z_3 = 0 \\ -2z_1 - z_2 - z_3 = 0 \\ 2z_1 + z_2 + z_3 = 0 \end{cases} \text{ and } \begin{cases} z_1 = x_1 + x_2 + x_3 \\ z_2 = -2x_1 - x_2 - x_3 \\ z_3 = 2x_1 + x_2 + x_3 \end{cases}$$

one system

(11)

$$\left(\begin{array}{cccccc|cccccc} 1 & -2 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & -2 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} I-II \\ \\ III-II \\ IV+II \\ \\ \\ \end{array} \sim$$

$$\sim \left(\begin{array}{cccccc|cccccc} 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & -2 & 2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ \\ IV-II \\ V-II \\ \\ \end{array} \sim$$

$$\sim \left(\begin{array}{cccccc|cccccc} 0 & -1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 2 & -1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\sim \begin{array}{c} \left(\begin{array}{cccccc|cccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{array}\right) \\ \begin{array}{cccccccccccc} & & & & & & x_1 & x_2 & x_3 & z_1 & z_2 & z_3 \end{array} \end{array}$$

$$y = c_1\left[\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} t\right] e^t + c_2\left[\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} t\right] e^t$$

(12) The general solution is given by

$$y_1 = C_2 e^t + C_3 e^{2t}$$
$$y_2 = -(C_1 + C_2) e^t - C_2 t e^t - 2 C_3 e^{2t}$$
$$y_3 = C_1 e^t + C_2 t e^t + 2 C_3 e^{2t}.$$