1 N25. Uniqueness and existence of solutions.
Higher order linear equations

1. Homogeneous non linear equations

An equation which can be written as

$$y' = g\left(\frac{y}{x}\right) \qquad (25.1)$$

is called homogeneous, e.g.

$$y' = \frac{y + x e^{-\frac{y}{x}}}{x} = \frac{y}{x} + e^{-\frac{y}{x}}.$$

Here, $g(u) = u + e^{-u}$.

We find a solution to (25.1) as

$$y(x) = u(x) \cdot x. \longrightarrow \text{into } (25.1)$$

$$u'x + u = g(u)$$

or

$$u' = \frac{1}{x}\left(g(u) - u\right) \text{ - separable eq.}$$

Ex 25.1

$$y' = \frac{y + x e^{-\frac{y}{x}}}{x} \qquad (25.2)$$

substitution $y = ux$ into (25.2) gives

$$u'x + u = \frac{ux + x e^{-\frac{ux}{x}}}{x} = u + e^{-u}$$

$$e^{u} u' = \frac{1}{x}$$

$$e^{u} \frac{du}{dx} = \frac{1}{x}$$

$$e^{u} du = \frac{dx}{x}$$

$$\int e^u \, du = \int \frac{dx}{x}$$

$$e^u = \ln|x| + C$$

$$u = \ln(\ln|x| + C)$$

and

$$y = x \ln(\ln|x| + C).$$

2. Existence and uniqueness of solutions

Th. 25.1 (Peano) Let $f = f(x,y)$ be continuous in $(a,b) \times (c,d)$ and $x_0 \in (a,b)$, $y_0 \in (c,d)$. Then there exists $\varepsilon > 0$ such that the initial value problem

$$\begin{aligned} y' &= f(x,y) \\ y(x_0) &= y_0 \end{aligned} \qquad (25.3)$$

has a solution on $[x_0 - \varepsilon, x_0 + \varepsilon]$

Th 25.2 (Picard - Lindelöf) Let $f = f(x,y)$ be

a) uniformly Lipschitz continuous in y

$\exists\, L > 0$ s.t. $|f(x,y_1) - f(x,y_2)| \leq L|y_1 - y_2|$

b) continuous in x.

Then there exists $\varepsilon > 0$ s.t. the initial value problem (25.3) has a unique solution on $[x_0 - \varepsilon, x_0 + \varepsilon]$.

③ 3. Higher order linear equation with constant coefficients.

We consider an equation

(25.4) $a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = F(x)$,

$a_0 \neq 0$

Def 25.1 • Equation (25.4) is called a higher order linear equation with constant coefficients.

• If $F = 0$ then the equation is said to be homogeneous.

In this section we will consider only homogeneous equations.

We will find solutions in the form

$$y = ce^{\lambda x} \qquad (25.5)$$

Substituting (25.5) into (25.4) (with $F=0$), we obtain

$$a_0 \lambda^n c e^{\lambda x} + a_1 \lambda^{n-1} c e^{\lambda x} + \dots + a_{n-1} \lambda c e^{\lambda x} + a_n c e^{\lambda x} = 0$$

Hence if λ satisfies

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n = 0, \qquad (25.6)$$

(4)

then (25.5) solves homogeneous equation (25.4).

<u>Def 25.2</u> The equation

$$a_0\lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n = 0$$

is called the <u>characteristic polynomials</u> of $Ly := a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0$. (25.6)

<u>Ex 25.2</u> Find solutions of

$$y''' - 6y'' + 11y' - 6y = 0$$

To find all solutions, we consider the characteristic polynomial of the corresponding equation

$$p(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

We can write

$$p(\lambda) = (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

so, $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

Then
$$y_1 = C_1 e^{\lambda x} = C_1 e^x$$
$$y_2 = C_2 e^{\lambda_2 x} = C_2 e^{2x}$$
$$y_3 = C_3 e^{\lambda_3 x} = C_3 e^{3x}$$

are solutions to the equation.

$y = C_1 e^{x} + C_2 e^{2x} + C_3 e^{3x}$ - solution to the equation.

Let $y_1, \dots, y_n$ be some functions.
if $y_1, \dots, y_n$ are solutions to (25.6), then $y = C_1 y_1 + \dots + C_n y_n$ is also a solution to $Ly = 0$.

Def 25.3 We say that $\{y_1, \dots y_n\}$ is a fundamental system of solutions to

$$Ly = 0,$$

if every solution of $Ly = 0$ can be written as a linear combination of $\{y_1, \dots, y_n\}$, i.e.

$$y = C_1 y_1 + \dots + C_n y_n$$

for some constants $C_1, \dots, C_n$.

Def 25.4 We say that $\{y_1, \dots, y_n\}$ is linearly independent if the equality

$$C_1 y_1(x) + \dots + C_n y_n(x) = 0 \quad \forall x$$

implies $C_1 = \dots = C_n = 0$.

Th 25.3 A set $\{y_1, \dots, y_n\}$ of n solutions to $Ly = 0$ is a fundamental system iff it is linearly independent.

Remark 25.1 Any solution to

$$y''' - 6y'' + 11y' - 6y = 0$$

can be written in the form

$$y = C_1 e^{x} + C_2 e^{2x} + C_3 e^{3x}$$

– such a solution is called <u>general solution</u>

Let us come back to solving the equation

$$Ly = 0.$$

Let its characteristic polynomial can be written

$$p(\lambda) = (\lambda - \lambda_1)^{k_1} \dots (\lambda - \lambda_m)^{k_m}$$

Consider the cases (omit indexes of λ_j and k_j)

1) $\lambda \in \mathbb{R}$, $k = 1$ (case of Ex. 25.2)

$y = e^{\lambda x}$ – solution

2) $\lambda \in \mathbb{R}$, $k > 1$

$y_1 = e^{\lambda x}$, $y_2 = x e^{\lambda x}$, $\dots$, $y_k = x^{k-1} e^{\lambda x}$

– solutions

3) $\lambda = a \pm ib$, $k = 1$

$$y_1 = e^{ax} \cos bx, \quad y_2 = e^{ax} \sin bx$$

– solutions

4) $\lambda = a \pm ib$, $k > 1$

$$y_1 = e^{ax} \cos bx, \quad y_2 = e^{ax} \sin bx$$

$$y_3 = x e^{ax} \cos bx, \quad y_3 = x e^{ax} \sin bx$$

$$y_{2k-1} = x^{k-1} e^{ax} \cos bx, \quad y_{2k} = x^{k-1} e^{ax} \sin bx$$

– solutions.

Ex 25.3 Find a solution to

$$y'' + 4y = 0$$

$$\lambda^2 + 4 = 0$$

$$\lambda = \pm 2i$$

$$y = C_1 \cos 2x + C_2 \sin 2x$$

– general solution to the equation.

Ex 25.4

$$y^{(4)} + 2y''' + 2y'' = 0$$

$$p(\lambda) = \lambda^4 + 2\lambda^3 + 2\lambda^2 = 0$$

$$\lambda^2(\lambda^2 + 2\lambda + 2) = 0$$

$\lambda_1 = 0$ or $\lambda^2 + 2\lambda + 2 = 0$

$$D = 4 - 4\cdot 2 = -4 = (2i)^2$$

$$\lambda_2 = \frac{2+2i}{2} = 1+i$$

$$\lambda_3 = 1-i$$

$$p(\lambda) = (\lambda - 0)^2(\lambda - (1+i))(\lambda - (1-i))$$

$$y = C_1 e^{0x} + C_2 x e^{0x} + C_3 e^{1x}\cos x + C_4 e^{1x}\sin x =$$

$$= C_1 + C_2 x + e^x(C_3\cos x + C_4 \sin x).$$

– general solution to the equation.

Ex 25.4 Solve the initial value problem

$$y'' + 4y = 0 \qquad \text{(see Ex. 25.3)}$$

$$\begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases} \qquad (25.7)$$

$y = C_1 \cos 2x + C_2 \sin 2x$ – general solution

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Find C_1, C_2 from (25.7).

$$y(0) = C_1 \cdot 1 + C_2 \cdot 0 = C_1 = 0$$

$$y'(x) = C_2 \cdot 2\cos 2x$$

$$y'(0) = C_2 \cdot 2 = 1 \quad \Rightarrow \quad C_2 = \frac{1}{2}.$$

$y(x) = \frac{1}{2}\sin 2x$ – solution to the initial value problem.