N 24 First order differential equations.

1. Separable diff. equations

Def 24.1 A differential equation of the form

$$y' = f(y)\, g(x) \qquad (24.1)$$

is called <u>separable</u>

In this section we describe the method of solving such an equation.

First we rewrite (24.1) in the form

$$\underbrace{\frac{1}{f(y)}}_{= h(y)} y' = g(x) \qquad \text{if } f(y) \neq 0.$$

So, we have

$$h(y)\, y' = g(x)$$

Next, let G and H be antiderivatives of g and h, that is

$$H'(y) = h(y), \quad G'(x) = g(x)$$

Then from the chain rule

$$\frac{d}{dx} H(y(x)) = H'(y(x))\, y'(x) = h(y)\, y'(x).$$

Thus,

$$\frac{d}{dx} H(y(x)) = \frac{d}{dx} G(x)$$

Integrating both sides of this equation, we have

some constant

$$H(y(x)) = G(x) + C \qquad (24.2)$$

Consequently, any differentiable function $y = y(x)$ that satisfies (24.2) is a solution to diff. eq. (24.1).

To solve (24.1) we need to find only antiderivatives of h and g.

<u>Ex 24.1</u> $y' = x(1+y^2)$

$$\frac{1}{1+y^2}\frac{dy}{dx} = x$$

$$\frac{dy}{1+y^2} = x\,dx$$

$$\int \frac{dy}{1+y^2} = \int x\,dx$$

this just is a convenient way to find H, G.

③

$$\arctan y = \frac{x^2}{2} + C$$

Hence

$$y = \tan\left(\frac{x^2}{2} + C\right)$$

<u>Ex 24.2</u>

$$y' = \frac{2x+1}{5y^4+1}$$

$$\frac{dy}{dx} = \frac{2x+1}{5y^4+1}$$

$$(5y^4+1)\,dy = (2x+1)\,dx$$

$$\int (5y^4+1)\,dy = \int (2x+1)\,dx$$

$$y^5 + y = x^2 + x + C$$ - implicit solution.

<u>Remark 24.1</u> Deviding (24.1) by $f(y)$ we can lose some solutions, namely, the constant solutions $y(x) = y_0$, where $f(y_0) = 0$.

<u>Ex 24.3</u> Find all solutions of

$$y' = 2xy^2 \quad : y^2, \quad y^2 \neq 0$$

$$\frac{1}{y^2}\frac{dy}{dx} = 2x$$

$$\frac{dy}{y^2} = 2x\,dx$$

$$\int \frac{dy}{y^2} = \int 2x\,dx$$

$$-\frac{1}{y} = x^2 + C$$

This is equivalent to

$$y = -\frac{1}{x^2+C} \qquad (24.3)$$

Note that $y=0$ is also a solution. Moreover, it can not be written in form (24.3)

<u>Ex 24.4</u> Initial value problem:

$$y' = 2xy^2$$

$$y(0) = 1$$

First, we find all solutions of the equation

$$y = -\frac{1}{x^2+C}.$$

Then find C from the initial condition

$$y(0) = -\frac{1}{0^2+C} = -\frac{1}{C} = 1 \Rightarrow C = -1$$

$$y(x) = -\frac{1}{x^2-1} = \frac{1}{1-x^2}, \quad x \in (-1, 1).$$

2. Linear first order diff. eq.

Def 24.2 • A first order diff. eq. is said to be linear if it can be written as

$$y' + p(x)\,y = f(x) \qquad (24.4)$$

• If $f = 0$, then (24.4) is called homogeneous. Otherwise, it is called non homogeneous.

We first consider the case $f=0$:

$$y' + p(x)\,y = 0. \qquad (24.5)$$

We solve (24.5) using the method of separation of variables from the previous section.

$$y' = -y\,p(x)$$

$$\frac{dy}{dx} = -y\,p(x)$$

$$\int \frac{dy}{y} = -\int p(x)\,dx$$

$$\ln|y| = -\underset{\underset{\int p(x)dx}{\parallel}}{P(x)} + \overset{\text{constant}}{k}$$

Hence

$$y = e^{k} e^{-P(x)}$$

$$y = -e^{k} e^{-P(x)}$$

⑥

Consequently, we can write solutions of (24.5) as

$$y = Ce^{-P(x)} = Ce^{-\int p(x)dx} \quad (24.6)$$

where $C \in \mathbb{R}$. Note, that for $C = 0$
$y = 0$ is also a solution to (24.5).

<u>Def 24.3</u> (24.6) is called the general solution to <u>homogeneous equation</u> (24.5)

In order solve (24.4) we assume, that the constant C in (24.6) depends on x, that is, $C = C(x)$

So $$y = C(x) e^{-P(x)} \quad (24.7)$$

Then, we substitute (24.7) into (24.4).

Compute $$y' = C'(x) e^{-P(x)} - C(x) e^{-P(x)} \underbrace{P'(x)}_{= p(x)}$$

consequently,

$$C'(x) e^{-p(x)} - C(x) e^{-P(x)} p(x) + \underbrace{C(x) e^{-P(x)} p(x)}_{= d(x)}$$

So, $$C'(x) = d(x) e^{P(x)}$$ and

$$C(x) = \int d(x) e^{P(x)} dx + C_1.$$

7

We have obtain

$$y(x) = \underbrace{\left(\int q(x)\, e^{P(x)}\,dx\right) e^{-P(x)}}_{\text{partial solution to (24.4)}} + \underbrace{C_1\, e^{-P(x)}}_{\text{general solution to homogeneous eq. (24.5)}}$$

<u>Remark 24.1</u> General solution to (24.4) can be written as the sum of a solution to (24.4) (partial solution) and general solution to the homogeneous equation (24.5). Indeed, if $y = y(x)$ is a solution to (24.4) and $y_1 = y_1(x)$ is another solution to (24.4). Then

$$y_1(x) = y(x) + \underbrace{y_1(x) - y(x)}_{\text{solution to (24.5)}}$$

<u>Ex 24.5</u> $y' - 2xy = e^{x^2}$

a) First solve:

$$y' - 2xy = 0$$

$$y = Ce^{-\int(-2x)dx} = Ce^{x^2}$$

b) Variation of parameters:

$$y = C(x)\, e^{x^2} \longrightarrow \text{into the equation}$$

$$C'(x)e^{x^2} + 2xC(x)e^{x^2} - \cancel{2xC(x)e^{x^2}} = e^{x^2}$$

$C'(x) = 1$, Hence

$$C(x) = \int 1\,dx = x + C_1$$

we have obtain

$$y(x) = (x + C_1)e^{x^2} = xe^{x^2} + C_1 e^{x^2}$$

3. Transformation of non linear equations

a) Bernoulli equation

$$(24.8) \qquad y' + p(x)y = f(x)y^r, \quad \text{where } r \in \mathbb{R} \text{ and } r \neq 0,\ r \neq 1.$$

Let y_1 be a nontrivial solution to

$$y_1' + p(x)y_1 = 0.$$

Then we find a solution to (24.8) as

$$y = u\,y_1,$$

where u is some function. So substituting $y = u y_1$ into (24.8), we obtain

$$u'y_1 + u\underbrace{(y_1' + p(x)y_1)}_{=0} = f(x)(uy_1)^r$$

which is equivalent to separable equation.

9

Ex 24.6 $\quad y' - y = xy^2$ — Bernoulli eq.

1) Firs solve

$$y' - y = 0$$

$$y = Ce^{-\int(-1)dx} = Ce^{x}.$$

Take $\quad y_1(x) = e^{x}.$

2) Find y as $y(x) = u(x)\, e^{x}$ ⟶ into the eq.

$$u'(x)e^{x} + \cancel{u(x)\,e^{x}} - \cancel{u(x)\,e^{x}} = x \cdot u^2(x)\, e^{2x}$$

$$u' = u^2 \cdot x\, e^{x} \quad \text{— separable eq.}$$

$$\frac{du}{dx} = u^2\, x e^{x}$$

$$\int \frac{du}{u^2} = \int x e^{x}\, dx$$

$$-\frac{1}{u} = \int x\, de^{x} = xe^{x} - \int e^{x} dx =$$

$$= xe^{x} - e^{x} + C$$

So, $\quad u = -\dfrac{1}{(x-1)e^{x} + C}$

Hence

$$y = -\frac{e^{x}}{(x-1)e^{x} + C} = -\frac{1}{x-1+Ce^{-x}}$$