Differential equations.

N 23 Basic concept of differential equations.

1. Models which leads to diff. eq.

a) Population growth and Decay

Let $P(t)$ denote a number of members of a population (people in a given country, bacteria in a laboratory culture).
For simplification of the mathematical model we assume that $P(t)$ can be any positive number (not necessarily integer).

Let a be the rate of change of population (per unit time per individual). Then

$$P'(t) = a P(t)$$

b) Spred of epidemics

Let the number of people infected changes at a rate proportional to the product of the number of people already infected and the number of people who are suspectible, but not yet infected.

②

If S denotes the total number of people and $I(t)$ denotes the number of infected people at time t, then

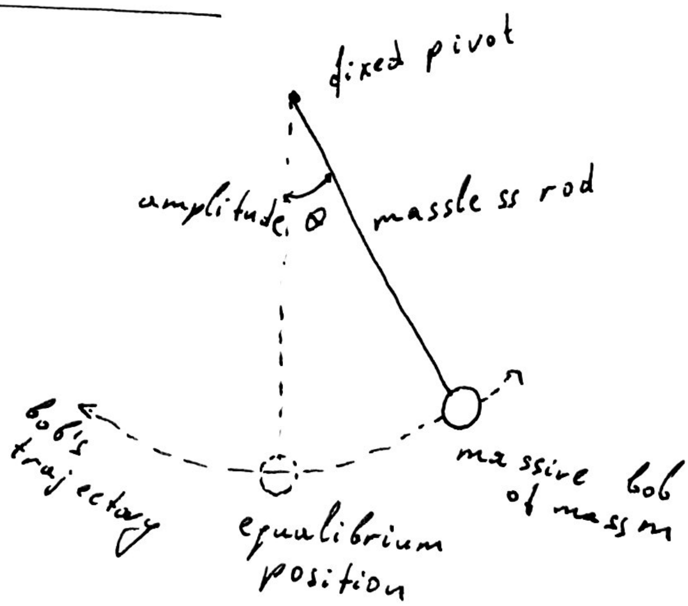
$$I'(t) = r\, I(t)(S - I(t)),$$

where r is a positive constant.
If at the initial time $t=0$ there were I_0 infected people, then we can add to the equation the condition.

$$I(0) = I_0$$

c) <u>Simple gravity pendulum</u>

We assume also that the motion of the bob occurs in two dimensions and the motion does not lose energy to friction or air resistance.



By Newton's second law

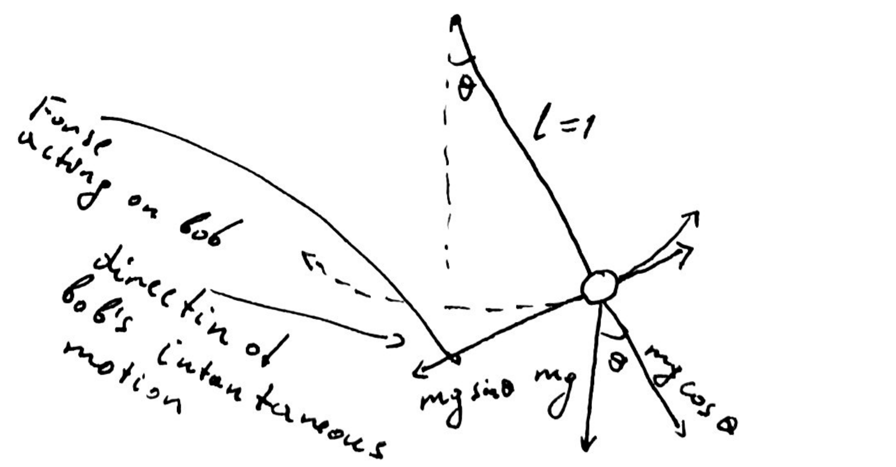
$$F = m\underbrace{a}_{\text{acceleration}}$$

Thus

$$-mg\sin\theta = ma$$

So $a = -g\sin\theta$

So, $\theta'' + g\sin\theta = 0$



if the oscilation's amplitude is small, then we can assume that $\sin\Theta \approx \Theta$.

So, we obtain the equation

$$\Theta'' + g\,\Theta = 0$$

2. Basic definitions

Def. 23.1 • A differential equation is an equation that contains one or more derivatives of an unknown function.

• The order of a differential eq. is the order of the highest derivative that it contains.

• The diff. eq. is called an ordinary differential equation if it involves an unknown function of only one variable.

Ex 23.1 a) $\frac{dy}{dx} - x^2 = 0$ or $y' - x^2 = 0$ (first order)

b) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2x$ (second order)

or

$$y'' + 2y' + y = 2x$$

c) $y^{(n)} = x - \sin y$ (n-th order)

We will usually consider diff. eq. that can be written as

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}) \qquad (23.1)$$

Def 23.2 A solution of diff. eq. (23.1) is a function $y = y(x)$ defined on some open interval (a, b), n times differentiable and

$$y^{n}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x))$$

for all $x \in (a, b)$.

Def 23.3 • The graph of a solution of a diff. eq. is called a solution curve.

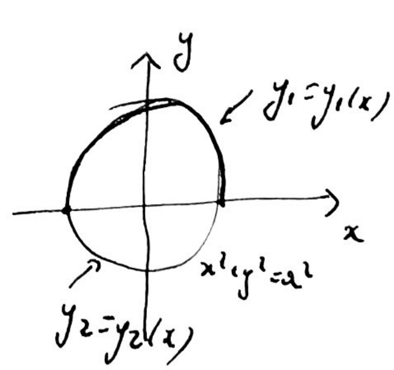
• A curve C is said to be an integral curve of a diff. eq. if every function $y = y(x)$ whose graph is a segment of C is a solution of the diff. eq.

Ex 23.2 The circle

$$x^2 + y^2 = a^2$$

is an integral curve of

$$y' = -\frac{x}{y} \qquad (23.2)$$



Indeed, $y_1 = \sqrt{a^2 - x^2}$ solves (23.2)

$$y_1' = -\frac{2x}{2\sqrt{a^2-x^2}} = -\frac{x}{\sqrt{a^2-x^2}} \qquad \forall x \in (-a, a)$$

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The same for $y_2 = -\sqrt{a^2 - x^2}$, $x \in (-a, a)$.
But $x^2 + y^2 = a^2$ is not a solution curve, since it is not a graph of some function $y = y(x)$.

<u>Ex 23.3</u> Let us find solutions of the equation

$$y'' = e^x \qquad (23.3)$$

Integrating (23.3), we obtain

$$y' = \int e^x \, dx = e^x + C_1$$

Integrating again

$$y = \int (e^x + C_1)\, dx = e^x + C_1 x + C_2$$

where C_1, C_2 some constants.

3. Initial Value Problem.

In Ex 23.3, one can see that diff. eq. (23.3) has infinitely many solutions depending on C_1, C_2. So we will often be interested in finding a solution of a diff. eq. that satisfies some additional conditions.

So, we will find solutions to the eq.

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}) \qquad (23.1)$$

which satisfies

$$y(x_0) = p_0,\ y'(x_0) = p_1, \dots, y^{(n-1)} = p_{n-1} \quad (23.4)$$

this conditions are called <u>initial conditions</u>

The problem of finding solutions to (23.1) which satisfies (23.4) is called an initial value problem.

<u>Ex 23.3</u> We consider the initial value problem

$$y'' = e^x$$

$$\begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$$

The general solution is

$$y = e^x + C_1 x + C_2$$

Find C_1, C_2: $y(0) = \underbrace{e^0}_{1} + C_1 \cdot 0 + C_2 = 1 \Rightarrow C_2 = 0$

$$y'(0) = e^0 + C_1 = 0 \Rightarrow C_1 = -1$$

$y = e^x - x$ is a solution of the initial value problem.

Under „nice" conditions of f in (23.1) the initial value problem has only one solution.

Ex 23.4
$$y' = 2\sqrt{y} \qquad (23.5)$$
$$y(0) = 0$$

One can check that $y(x) = 0$ and
$y(x) = x^2$, $x \geq 0$
are two different solutions to the initial value problem (23.5).

Remark 23.1 If f is continuous, then
$$y' = f(x)$$
$$y(x_0) = y_0$$
admits solution $y(x) = y_0 + \int_{x_0}^{x} f(t)\,dt$.

4. Directional fields for first order diff. eq.

We consider the equation
$$y' = f(x,y) \qquad (23.6)$$
We discuss the „graphical method of solving the equation (23.6)"

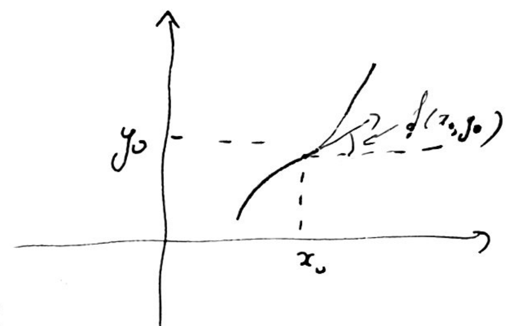
We recall that $y = y(x)$ is a solution to (23.6), if

$$y'(x) = f(x, y(x))$$

for all x from sone interval.

So, the slope of the ingegral curve of (23.6) through a point (x_0, y_0) is given by the number $f(x_0, y_0)$.

If f is defined on a set R, we can construct a direction field for (23.6) in R by drawing a short line (or vector) segment through each point (x, y) in R with slope $f(x, y)$.



Then integral curves of the equation (23.6) are continuous curve tangent to the vectors in directional field.