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N 22. Conditional local extrema

1. Some exercises.

We consider some exercises which leads to the finding of conditional extrema.

Exercise 22.1.

Find the point in the plane

$$3x + 4y + z = 1$$

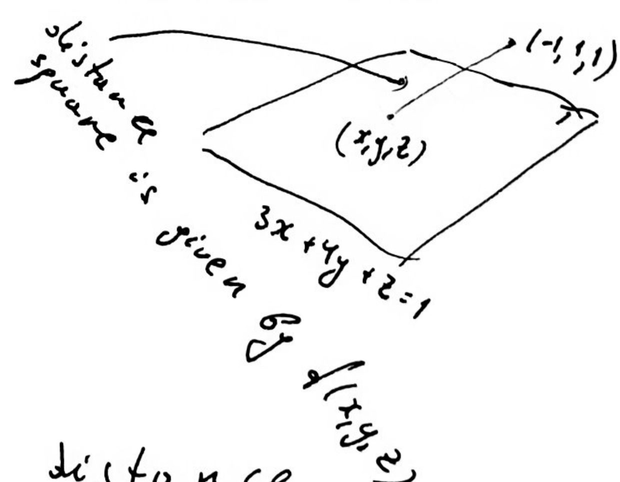
closest to $(-1, 1, 1)$.

To find such a point, we have to minimize

$$f(x,y,z) = (x+1)^2 + (y-1)^2 + (z-1)^2$$

subject to the constraint

$$3x + 4y + z = 1.$$



Exercise 22.2 Find the distance between two curves in $\mathbb{R}^2$ given by:

$x^2 + 2y^2 = 1$ (ellipse)

$x + y = 4$ (line)

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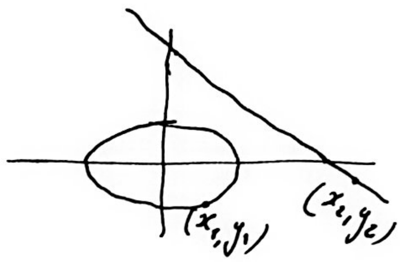
To find the distance, we have to compute the minimum of

$$d(x_1, y_1, x_2, y_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

subject to the constraint

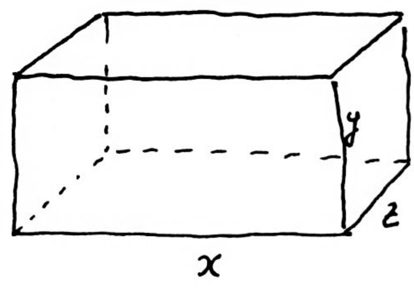


$$\begin{cases} x_1^2 + 2y_1^2 = 1 \\ x_2 + y_2 = 4, \end{cases}$$

and thet take the squareroot.

<u>Exercise 22.3</u> For which size of an open rectangular bath of the volume V it has the smallest surface area.

We have to minimize



$$d(x,y,z) = xz + 2zy + 2xy$$

under the constrain

$$xyz = V$$

<u>The general formulation of the problem:</u>

Let $f : D \to \mathbb{R}$, $g_i : D \to \mathbb{R}$, $i = 1, \dots, m$, where $D \subseteq \mathbb{R}^d$ be an open set

Find a conditional local extrema of the function f subject to

$$g_1 = 0, \dots, g_m = 0.$$

③ 2. Method of Lagrange multipliers.

Definition 22.1 • Let $M = \{x \in D : g_1(x) = 0, \dots, g_m(x) = 0\}$

A point $x_0 \in D$ is called a conditional local maximum (minimum) of f subject to the constraint $g_1 = 0, \dots, g_m = 0$ if

$$\exists r > 0 \quad \forall x \in B_r(x_0) \cap M \quad f(x) \le f(x_0)$$
$$(f(x) \ge f(x_0))$$

• $x_0 \in D$ is a conditional local extrema of f if x_0 is a conditional local maximum or minimum.

Th 22.1 We assume that $m < d$, $f \in C^1(D)$, $g_i \in C^1(D)$, $i = 1, \dots, m$ and the matrix

$$\left(\frac{\partial g_i}{\partial x_j}(x_0) \right)_{i=1, j=1}^{m, d}$$

has rank m at a point x_0, which is a conditional local extrema of f subject to the constraint

$$g_1 = 0, \dots, g_m = 0.$$

Then there exist real numbers $\lambda_1, \dots, \lambda_m$ for which the point x_0 is a critical point of the function

$$F(x) = f(x) - \lambda_1 g_1(x) - \dots - \lambda_m g_m(x),$$

that is,

$$\frac{\partial F}{\partial x_j}(x_0) = 0, \quad \forall j = 1, \dots, d.$$

The following implementation of Th 22.1 is the method of <u>Lagrange multipliers</u>:

a) Find all solutions $x_1, \dots, x_d, \lambda_1, \dots, \lambda_m$ of the system of equations

$$\begin{cases} \frac{\partial F}{\partial x_j}(x) = 0, & j = 1, \dots, d \\ g_i(x) = 0, & i = 1, \dots, m \end{cases}$$

b) Determine which of the critical points are conditional extrema of f.
This can usually be done by physical or intuitive argument.

<u>Ex 22.1</u> We solve Exercise 22.1.

$$f(x, y, z) = (x+1)^2 + (y-1)^2 + (z-1)^2$$

$$g(x, y, z) = 3x + 4y + z - 1 = 0$$

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$$F(x,y,z) = (x+1)^2 + (y-1)^2 + (z-1)^2 - \lambda(3x+4y+z-1)$$

$$\begin{cases} \dfrac{\partial F}{\partial x} = 2(x+1) - 3\lambda = 0 & \Rightarrow \quad x = \dfrac{3\lambda}{2} - 1 \\ \dfrac{\partial F}{\partial y} = 2(y-1) - 4\lambda = 0 & \Rightarrow \quad y = 2\lambda + 1 \\ \dfrac{\partial F}{\partial z} = 2(z-1) - \lambda = 0 & \Rightarrow \quad z = \dfrac{\lambda}{2} + 1 \\ g(x,y,z) = 3x + 4y + z - 1 = 0 & \end{cases}$$

$$\frac{9\lambda}{2} - 3 + 8\lambda + 4 + \frac{\lambda}{2} + \cancel{1} - \cancel{1} = 0$$

$$\frac{26\lambda}{2} = -1 \quad \Rightarrow \quad \lambda = -\frac{1}{13}$$

So,

$$x = -\frac{3}{26} - 1 = -\frac{29}{26}$$

$$y = -\frac{2}{13} + 1 = \frac{11}{13}$$

$$z = \frac{1}{26} + 1 = \frac{27}{26}$$

Hence, the point $(x_0, y_0, z_0) = \left(-\frac{29}{26}, \frac{11}{13}, \frac{27}{26}\right)$ is the point of local minimum

(since, the function f must have the unique conditional local minimum due to the geometrical reason)

The distance is

$$\sqrt{(x_0+1)^2 + (y_0-1)^2 + (z_0-1)^2} = \frac{1}{\sqrt{26}}.$$

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Ex 22.2 We solve Exercise 22.2.

$$d(x_1, y_1, x_2, y_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$\begin{cases} g(x_1, y_1, x_2, y_2) = x_1^2 + 2y_1^2 - 1 = 0 \\ g(x_1, y_1, x_2, y_2) = x_2 + y_2 - 4 = 0 \end{cases}$$

Using the Lagrange multipliers method, we obtain:

$$F(x_1, y_1, x_2, y_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2 -$$

$$- \lambda_1 (x_1^2 + 2y_1^2 - 1) - \lambda_2 (x_2 + y_2 - 4)$$

$$\begin{cases} \dfrac{\partial F}{\partial x_1} = 2(x_1 - x_2) - 2\lambda_1 x_1 = 0 \\ \dfrac{\partial F}{\partial y_1} = 2(y_1 - y_2) - 4\lambda_1 y_1 = 0 \\ \dfrac{\partial F}{\partial x_2} = -2(x_1 - x_2) - \lambda_2 = 0 \\ \dfrac{\partial F}{\partial y_2} = -2(y_1 - y_2) - \lambda_2 = 0 \\ x_1^2 + 2y_1^2 - 1 = 0 \\ x_2 + y_2 - 4 = 0 \end{cases}$$

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$$\begin{aligned}
&\mathrm{I}+\mathrm{III}: \\
&\mathrm{II}+\mathrm{IV}:
\end{aligned}
\quad
\begin{cases}
-2\lambda_1 x_1 - \lambda_2 = 0 \\
-4\lambda_1 y_1 - \lambda_2 = 0 \\
x_1 - x_2 = -\dfrac{\lambda_2}{2} \\
y_1 - y_2 = -\dfrac{\lambda_2}{2} \\
x_1^2 + 2y_1^2 - 1 = 0 \\
x_2 + y_2 - 4 = 0
\end{cases}$$

$$\mathrm{I}-\mathrm{II}: \quad
\begin{cases}
4\lambda_1 y_1 - 2\lambda_1 x_1 = 0; \quad \lambda_1(2y_1 - x_1) = 0 \iff \lambda_1 = 0 \text{ or } x_1 = 2y_1 \\
-4\lambda_1 y_1 - \lambda_2 = 0 \\
x_1 - x_2 = -\dfrac{\lambda_2}{2} \\
y_1 - y_2 = -\dfrac{\lambda_2}{2} \\
x_1^2 + 2y_1^2 - 1 = 0 \\
x_2 + y_2 - 4 = 0
\end{cases}$$

1) if $\lambda_1 = 0 \overset{\mathrm{II}}{\Rightarrow} \lambda_2 = 0 \overset{\mathrm{III},\mathrm{IV}}{\Rightarrow} x_1 = x_2,\ y_1 = y_2$ that is impossible since the line and ellipse do not intersect.

2) $\begin{cases} x_1 = 2y_1 \\ x_1^2 + 2y_1^2 - 1 = 0 \end{cases} \Rightarrow 4y_1^2 + 2y_1^2 = 1 \Rightarrow y_1 = \pm\dfrac{1}{\sqrt{6}}$

$x_1 = \pm\dfrac{2}{\sqrt{6}}$

Since (x_1, y_1) belongs to the first quadrant $(x_1^\circ, y_1^\circ) = \left(\dfrac{2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}\right)$

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Next, $\overline{III} - \overline{IV}$: $x_1 - x_2 - y_1 + y_2 = 0 \Rightarrow$

$$y_2 - x_2 = y_1 - x_1 = -\frac{1}{\sqrt{6}}$$

Then, combining with $\overline{IV}$:

$$\begin{cases} y_2 - x_2 = -\frac{1}{\sqrt{6}} \\ x_2 + y_2 = 4 \end{cases}$$

So, $y_2 = 2 - \frac{1}{2\sqrt{6}}$

$x_2 = 2 + \frac{1}{2\sqrt{6}}$

So, $(x_2^\circ, y_2^\circ) = \left(2 + \frac{1}{2\sqrt{6}}, 2 - \frac{1}{2\sqrt{6}}\right)$.

At the point $(x_1^\circ, y_1^\circ, x_2^\circ, y_2^\circ)$ the function f takes a conditional local minimum. So,

$$\sqrt{\left(2 + \frac{1}{2\sqrt{6}} - \frac{2}{\sqrt{6}}\right)^2 + \left(2 - \frac{1}{2\sqrt{6}} - \frac{1}{\sqrt{6}}\right)^2} = \sqrt{2}\left(2 - \frac{3}{2\sqrt{6}}\right).$$

is the distance between two graphs