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N21. Extrema of functions of several variables

1. Necessary conditions of local extrema

Let $D \subseteq \mathbb{R}^d$ and $f: D \to \mathbb{R}$.

Def 21.1 • A point $x_0 \in D$ is called a local maximum (minimum) of f if there exists $r > 0$ such that

1) $B_r(x_0) \subseteq D$

2) $f(x) \leq f(x_0) \quad \forall x \in B_r(x_0)$

($f(x) \geq f(x_0)$ for local minimum)

• x_0 is a point of local extrema if it is a local minimum or maximum.

Th 21.1 Let x_0 be a local extrema of f and there exists $\nabla f(x_0)$. Then

$$\nabla f(x_0) = 0.$$

Proof We take the function

$$g_1(x_1) = f(x_1, x_2^0, \ldots, x_d^0).$$

Let x_0 be a local maximum.

Then, $g_1(x_1) \leq g_1(x_1^0) = f(x_0)$

for all $x_1 \in (x_1^0 - r, x_1^0 + r)$.

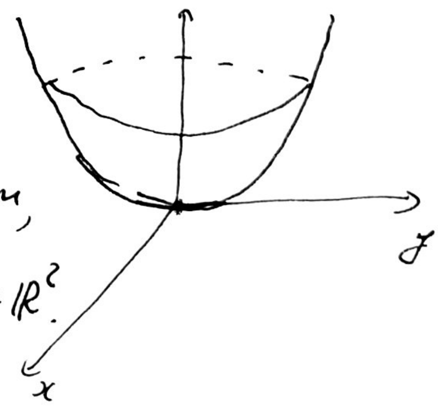
Hence x_1^0 is a local maximum of g_1.
By the Fermat theorem (see Th 11.2 Math 1)

$$g_1'(x_0) = \frac{\partial}{\partial x_1} f(x_0) = 0.$$

Similarly, $\frac{\partial f}{\partial x_k}(x_0) = 0$ for all $k=1,\dots,d$. ∎

<u>Def 21.2</u> Let x_0 be an inner point of D. If $\nabla f(x_0) = 0$, then x_0 is called a critical point of f.

<u>Ex. 21.1</u> Consider $f(x,y) = x^2 + y^2$, $(x,y) \in \mathbb{R}^2$.
The point $(0,0)$ is a local minimum of f.
It is also the global minimum, that is, $f(0,0) \le f(x,y)$ $\forall (x,y) \in \mathbb{R}^2$.



By Th 21.1

$$\nabla f(0,0) = 0.$$

Indeed, $\nabla f(x,y) = (2x, 2y)$. So, $\nabla f(0,0) = (2\cdot 0, 2\cdot 0) = (0,0)$.

<u>Remark 21.1</u> If x_0 is a critical point, then x_0 is not a local extrema in general.

Ex 21.2 $f(x,y) = x^2 - y^2$. The point $(0,0)$ is a critical point of f. Indeed,

$$\nabla f(x,y) = (2x, -2y) = (0,0) \text{ iff } (x,y) = (0,0).$$

But
$$f(0,0) \leq f(x,0) = x^2 \quad \forall x \in \mathbb{R}$$
$$f(0,0) \geq f(0,y) = -y^2 \quad \forall y \in \mathbb{R}.$$

So, $(0,0)$ is not a local extrema of f.

2. Sufficient conditions of local extrema.

Th 21.2 (Sufficient conditions of local extrema)

Let D be an open set and $x_0 \in D$. We assume that $f \in C^2(D)$ and x_0 is a critical point of f, i.e. $\nabla f(x_0) = 0$.

Then

a) if $\mathrm{Hess}_{x_0} f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=1}^{d}$ is positive definite (see Def 14.6), then x_0 is a local minimum of f.

b) if $\mathrm{Hess}_{x_0} f$ is negative definite, then x_0 is a local maximum of f.

c) if $\mathrm{Hess}_{x_0} f$ is indefinite, i.e.

$$\langle (\mathrm{Hess}_{x_0} f) u, u \rangle > 0, \quad \langle (\mathrm{Hess}_{x_0} f) v, v \rangle < 0$$

for some u and v,

then x_0 is not a local extrema of f.

④

We recall that a matrix $A=(a_{ij})_{i,j=1}^{d}$ is positive (negative) definite, if the bilinear form

$$B(u,u) = \langle Au, u\rangle = \sum_{i,j=1}^{n} a_{ij}\, u_i u_j$$

is positive (negative) definite, that is,

$$B(u,u) > 0 \quad \forall u \neq 0$$

$$(B(u,u) < 0 \quad \forall u \neq 0 \text{ for negative definite}).$$

<u>Proof of a)</u>. By the Taylor's formula (see Remark 20.2),

D

$$f(x) = f(x_0) + \langle \overbrace{\nabla f(x_0)}^{=0}, (x-x_0)\rangle +$$

$$+ \frac{1}{2} \langle \left(\mathrm{Hess}_{\tilde{x}} f\right)(x-x_0), (x-x_0)\rangle,$$

where $\tilde{x} = (1-\theta)x_0 + \theta x$ for some $\theta \in [0,1]$.

Since $\mathrm{Hess}_x f$ is continuous, because $f \in C^2(D)$, and positive definite at x_0, then $\mathrm{Hess}_x f$ is positive definite for all $x \in B_r(x_0)$, where r is some positive number. So,

$$f(x) = f(x_0) + \frac{1}{2} \overbrace{\langle \mathrm{Hess}_{\tilde{x}} f \underbrace{(x-x_0)}_{u}, \underbrace{(x-x_0)}_{u}\rangle}^{\geq 0} \geq$$

$\geq f(x_0)$ because $\mathrm{Hess}_{\tilde{x}} f$ is positive definite for all $x \in B_r(x_0)$

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Let $D \subseteq \mathbb{R}^2$, $f: D \to \mathbb{R}$.

We compute

$$\det(f''(x_0)) := \det \mathrm{Hess}_{x_0} f = \det \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x_0) & \frac{\partial^2 f}{\partial x \partial y}(x_0) \\ \frac{\partial^2 f}{\partial x \partial y}(x_0) & \frac{\partial^2 f}{\partial y^2}(x_0) \end{pmatrix} =$$

$$= \frac{\partial^2 f}{\partial x^2}(x_0) \frac{\partial^2 f}{\partial y^2}(x_0) - \left(\frac{\partial^2 f}{\partial x \partial y}(x_0) \right)^2 .$$

<u>Consequence 21.1</u> Let D be open in $\mathbb{R}^2$ and $x_0 \in D$. We assume that

$$\frac{\partial f}{\partial x_1}(x_0) = 0, \quad \frac{\partial f}{\partial x_2}(x_0) = 0 .$$

a) if $\frac{\partial^2 f}{\partial x^2}(x_0) > 0$ and $\det f''(x_0) > 0$, then x_0 is a local minimum

b) if $\frac{\partial^2 f}{\partial x^2}(x_0) < 0$ and $\det f''(x_0) > 0$, then x_0 is a local maximum

c) if $\det f''(x_0) < 0$, then x_0 is not a local extremum.

(6)

Ex 21.3 $f(x,y) = 2x^2 - y(y-1)^2, \quad (x,y) \in \mathbb{R}^2$

1) Find critical points:

$$\begin{cases} \frac{\partial f}{\partial x} = 4x = 0 \\ \frac{\partial f}{\partial y} = -(y-1)^2 - 2y(y-1) = -(y-1)(y-1+2y) = \\ \quad = -(y-1)(3y-1) = 0 \end{cases}$$

$$\left.\begin{array}{l} x=0,\ y=1 \\ x=0,\ y=\frac{1}{3} \end{array}\right\} \text{ critical points}$$

2) Check if critical points are local extrema, using sufficient conditions.

$$\frac{\partial^2 f}{\partial x^2} = 4 \qquad \frac{\partial^2 f}{\partial y^2} = -(3y-1) - 3(y-1) =$$
$$= -6y + 4$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\det f'' = 4 \cdot (-6y+4)$$

a) ~~Since~~ $\cancel{\frac{\partial^2 f}{\partial x^2}(0,1) = 4 > 0}$

Since $\det f''(0,1) = 4 \cdot (-6 \cdot 1 + 4) = -8 < 0$,

$(0,1)$ is not a local extrema. it is a saddle point

6) Since $\frac{\partial^2 f}{\partial x^2}(0, \frac{1}{3}) > 0$, and

$\det f''(0, \frac{1}{3}) = 4 \cdot (-6 \cdot \frac{1}{3} + 4) = 8 > 0$,

$(0, \frac{1}{3})$ is a local minimum.

Exercise 21.1 Find local extrema of the function $f(x,y) = (x+y)\, e^{-x^2-y^2}$.

Ex 21.4 $f(x,y,z) = x^2 + y^2 + z^2 + 2x + 4y - 6z + xy$

1) Find critical points

$$\begin{cases} \frac{\partial f}{\partial x} = 2x + y + 2 = 0 \\ \frac{\partial f}{\partial y} = 2y + x + 4 = 0 \quad \Big| \cdot (-2) + I \\ \frac{\partial f}{\partial z} = 2z - 6 = 0 \end{cases}$$

$$\begin{cases} -3y - 6 = 0 \\ 2y + x + 4 = 0 \\ z = 3 \end{cases} \qquad \begin{cases} y = -2 \\ x = 0 \\ z = 3 \end{cases}$$

$(0, -2, 3)$ - a critical point

(8) 2) check if $(0, -2, 3)$ is a local extrema

$$\text{Hess } f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

By Sylvester's criterion (ses Th 14.6)

$M_1 = 2 > 0$, $M_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0$

$$M_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8 + 0 + 0 - 0 - 0 - 2 = 6 > 0$$

$\text{Hess}_{(0,-2,3)} f$ is positive definite.

So $(0, -2, 3)$ is a local minimum.