(1)

N 20. Implicit function theorem and higher order derivatives

1. Implicit function theorem.

Let D be an open set in $\mathbb{R}^d$.

We denote by $\underline{C^1(D, \mathbb{R}^m)}$ the set of functions $f: D \to \mathbb{R}^m$ which are differentiable at each point $x \in D$ and $f': D \to \mathbb{R}^{m \times d}$ is continuous.

Th 20.1 Let D be an open set in $\mathbb{R}^d$, $f: D \to \mathbb{R}^d$, $x_0 \in D$ and $y_0 = f(x_0)$. Assume that following conditions hold

1) $f \in C^1(D, \mathbb{R}^d)$

2) $\det f'(x_0) \neq 0$.

Then there exists an open set $G \subseteq D$ which contains x_0, and a ball $B_r(y_0)$ such that

a) $f: G \to B_r(y_0)$

is a bijective map

b) the inverse map $g = f^{-1}: B_r(y_0) \to G$ belongs to $C^1(B_r(y_0), \mathbb{R}^d)$

(2)

c) $\forall y \in B_\varepsilon(y_0)$

$$g'(y) = \left(f'(g(y))\right)^{-1}.$$

Th 20.2 (Implicit function theorem).

Let G be an open set in $\mathbb{R}^{d+m}$, $(x_0, y_0) \in G$, $x_0 \in \mathbb{R}^d$, $y_0 \in \mathbb{R}^m$.
Assume that $F: G \to \mathbb{R}^m$ satisfies the following properties:

1) $F(x_0, y_0) = 0$;
2) $F \in C^1(G, \mathbb{R}^m)$
3) $\det F'_y(x_0, y_0) \neq 0$

← derivative of F with respect to y

Then there exists a ball $B_\varepsilon(x_0) \subset \mathbb{R}^d$ and a unique function $h: B_\varepsilon(x_0) \to \mathbb{R}^m$, $h \in C^1(B_\varepsilon(x_0), \mathbb{R}^m)$ such that

a) $h(x_0) = y_0$

b) $F(x, h(x)) = 0 \quad \forall x \in B_\varepsilon(x_0)$

c) $h'(x) = -\left(F'_y(x, h(x))\right)^{-1} \cdot F'_x(x, h(x)).$

Remark 20.1 Part c) follows from b) and the chain rule. Indeed,

$$0 = \left(F(x, h(x))\right)' = F'_x(x, h(x)) + F'_y(x, h(x)) \cdot h'(x).$$

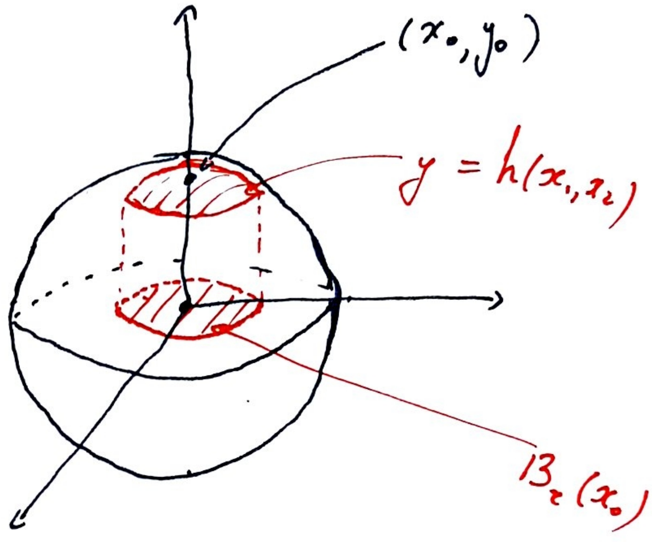
③

Ex. 20.1 Let $F(x_1, x_2, y) = x_1^2 + x_2^2 + y^2 - 1$

The equality $F(x_1, x_2, y) = x_1^2 + x_2^2 + y^2 - 1$ defines the sphere in $\mathbb{R}^3$

Take $x_0 = (0, 0)$

$y_0 = 1$



Then

1) $F(x_0, y_0) = 0$

2) $F \in C'(G, \mathbb{R})$

3) $F_y'(x_0, y_0) = 2y_0 = 2 \neq 0$

So, there exists a ball $B_r(x_0)$ and the unique function $y = h(x_1, x_2) : B_r(x_0) \to \mathbb{R}$

(In this example $h(x_1, x_2) = \sqrt{1 - x_1^2 - x_2^2}$)

such that

a) $y_0 = h(x_1^\circ, x_2^\circ)$

b) $F(x_1, x_2, h(x_1, x_2)) = 0 \quad (x_1^2 + x_2^2 + h^2(x_1, x_2) - 1 = 0)$

c) $F_x'(x, y) = (2x_1, 2x_2)$

$$h'(x_1, x_2) = -\frac{1}{2y}(2x_1, 2x_2) = \left(-\frac{x_1}{h(x_1, x_2)}, -\frac{x_2}{h(x_1, x_2)}\right)$$

$\forall (x_1, x_2) \in B_r(x_0)$

2. Higher order derivatives.

Let D be an open set in $\mathbb{R}^d$ and $f: D \to \mathbb{R}$.

Def 20.1 The second-order partial derivative of f at point $x_0 \in D$ is called

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)(x_0) =: \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)$$

if it exists.

Ex. 20.2 $f(x, y) = x^2 y + y$

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial^2 f}{\partial x^2} = 2y, \quad \frac{\partial^2 f}{\partial y \partial x} = \underline{2x}$$

$$\frac{\partial f}{\partial y} = x^2 + 1, \quad \frac{\partial^2 f}{\partial y^2} = 0, \quad \frac{\partial^2 f}{\partial x \partial y} = \underline{2x}$$

Th 20.3 (Schwarz's Theorem) If $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are continuous on D, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Ex. 20.3. Let $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & x^2+y^2>0 \\ 0, & x=y=0 \end{cases}$

We compute $\frac{\partial^2 f}{\partial x \partial y}(0,0)$ and $\frac{\partial^2 f}{\partial y \partial x}(0,0)$

(5)

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(0+t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{(xy(x^2-y^2))'_x(x^2+y^2) - xy(x^2-y^2)(x^2+y^2)'_x}{(x^2+y^2)^2} =$$

$$= \frac{\left(y(x^2-y^2)+2x^2y\right)(x^2+y^2) - 2x^2y(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) = \lim_{t\to 0} \frac{\frac{\partial f}{\partial x}(0,0+t) - \frac{\partial f}{\partial x}(0,0)}{t} =$$

$$= \lim_{t\to 0} \frac{(t(0-t^2)+0)(0+t^2)-0}{(0+t^2)^2\cdot t} = \lim_{t\to 0}(-1) = -1$$

we note that $f(x,y) = -f(y,x)$.

Thus, $\frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial f(y,x)}{\partial y}\right)$

deriv. w.r.t. the secont variable

derivative with respect to first variable

So, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) = -(-1) = 1$,

So, $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$.

Def 20.2 The matrix

$$\mathrm{Hess}_{x_0} f := f''(x_0) := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) \right)_{i,j=1}^{d}$$

is called the <u>second order derivative of f at point x_0</u> or the <u>Hessian matrix of f</u>

Ex 20.4 $f(x,y) = x e^y + y$, $\frac{\partial f}{\partial x} = e^y$

$\frac{\partial f}{\partial y} = x e^y + 1$

$$f''(x,y) = \begin{pmatrix} 0 & e^y \\ e^y & x e^y \end{pmatrix}$$

By the same way, one can introduce the n^{th}-order derivative of f:

$$\frac{\partial^n f}{\partial x_{i_1} \dots \partial x_{i_n}}(x_0) = \frac{\partial}{\partial x_{i_1}}\left(\frac{\partial}{\partial x_{i_2}} \cdots \left(\frac{\partial f}{\partial x_{i_n}} \right) \dots \right)(x_0)$$

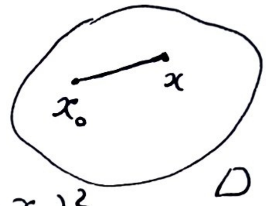
Def 20.3 We define $C^n(D)$ as a class of functions $f: D \to \mathbb{R}$ such that there exists $\frac{\partial^n f}{\partial x_{i_1} \dots \partial x_{i_n}}$ and is continuous on D for every $i_1, \dots, i_n = 1, \dots, d$.

⑦

3. Taylor's theorem

Th 20.4 We assume that $f \in C^n(D)$. Let x_0, $x \in D$ and for all $\theta \in [0,1]$

$$(1-\theta)x_0 + \theta x \in D$$



Then

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2$$
$$+ \frac{1}{(n-1)!} f^{(n-1)}(x_0)(x-x_0)^{n-1} + \frac{1}{n!} f^{(n)}((1-\theta)x_0 + \theta x)(x-x_0)^n,$$

where θ is some point from $[0,1]$,

and $f^{(k)}(x_0)(x-x_0)^k = \sum_{i_1,\dots,i_k=1}^{d} \frac{\partial^k f(x_0)}{\partial x_{i_1} \dots \partial x_{i_k}} (x_{i_1}-x_{i_1}^0)\dots(x_{i_k}-x_{i_k}^0)$.

Remark 20.2 If $n=2$, then

$$f(x) = f(x_0) + \langle \nabla f(x_0), (x-x_0)\rangle + \frac{1}{2} \underbrace{\langle (\mathrm{Hess}_{\tilde{x}} f)(x-x_0), (x-x_0)\rangle}_{\text{bilinear form}}$$

where $\tilde{x} = (1-\theta)x_0 + \theta x$ and θ is some point from $[0,1]$.

Ex 20.5 $f(x,y) = \sin x \sin y$

$$\nabla f(x,y) = (\cos x \sin y, \sin x \cos y)$$

$$\mathrm{Hess}_{(x,y)} f = \begin{pmatrix} -\sin x \sin y & \cos x \cos y \\ \cos x \cos y & -\sin x \sin y \end{pmatrix} \left(= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ at } (0,0) \right)$$

$x_0 = (0,0)$

$$f(x,y) \approx 0 + 0 + \frac{1}{2}xy + \frac{1}{2}xy = xy.$$