(1)

№19 Differentiation of functions of several variables II.

1. Derivatives of real-valued functions.

Let $D \subseteq \mathbb{R}^d$ and $f: D \to \mathbb{R}$.

- We recall that f is differentiable at an inner point x_0 of D if there exists a linear map
$$L: \mathbb{R}^d \to \mathbb{R}$$
such that
$$f(x) - f(x_0) - L(x-x_0) = o(\|x-x_0\|), \quad x \to x_0$$
i.e.
$$\lim_{x \to x_0} \frac{f(x)-f(x_0)-L(x-x_0)}{\|x-x_0\|} = 0.$$

- The linear map L is called the differential of f at point x_0 and is denoted by $df(x_0)$

- By Riesz's representation theorem there exists $v = (v_1, \dots, v_d) \in \mathbb{R}^d$, such that
$$L(x) = \langle x, v \rangle = x_1 v_1 + \dots + x_n v_n$$

②

- The vector v equals the gradient $\nabla f(x_0)$ of f, where

$$\nabla f(x_0) = \left(\frac{\partial f}{\partial x_1}(x_0), \dots, \frac{\partial f}{\partial x_d}(x_0) \right).$$

- We also use the notation for the differential

$$df(x_0) = \frac{\partial f}{\partial x_1}(x_0)\, dx_1 + \dots + \frac{\partial f}{\partial x_d}(x_0)\, dx_d .$$

<u>Th 19.1</u> (Chain rule) Let $f: D \to \mathbb{R}$ be differentiable at x_0. Let also $x_k = x_k(t_1, \dots, t_m)$, such that $x_k^\circ = x_k(t_1^\circ, \dots, t_m^\circ)$ and $\exists \ \frac{\partial x_k}{\partial t_j}(t^\circ)$ $\forall k, j$. Then for the function $h(t) = f(x(t))$ there exists a partial derivatives

$$\frac{\partial h}{\partial t_j}(t_0) = \sum_{k=1}^{d} \frac{\partial f}{\partial x_k}(x_0) \frac{\partial x_k}{\partial t_j}(t_0).$$

Let $\ell = (\ell_1, \dots, \ell_d) \in \mathbb{R}^d$, $f: D \to \mathbb{R}$, x_0 - inner point of D

<u>Def 19.1</u> The limit

$$\frac{\partial f}{\partial \ell}(x_0) = \lim_{t \to 0} \frac{f(x_0 + t\ell) - f(x_0)}{t},$$

if it exists, is called the <u>directional derivative along the vector ℓ</u>

Th 18.2 Let f be differentiable at x_0. Then for any $l = (l_1, \dots, l_d) \in \mathbb{R}^d$ there exisds the directional derivative along l and

$$\frac{\partial f}{\partial l}(x_0) = \sum_{k=1}^{d} \frac{\partial f}{\partial x_k}(x_0) \cdot l_k = \langle \nabla f(x_0), l \rangle.$$

Proof The prood is similar to the proof of Th 18.1.

Th 19.3 Let f be differentiable at x_0, then

$$\max_{\|l\|=1} \frac{\partial f}{\partial l}(x_0) = \| \nabla f(x_0) \|,$$

moreover, max is attained by a vector with the same direction as $\nabla f(x_0)$.

Proof By the Cauchy - Schwarz inequality

$$\frac{\partial f}{\partial l}(x_0) = \langle \nabla f(x_0), l \rangle \le \| \nabla f(x_0) \| \underbrace{\|l\|}_{=1} = \| \nabla f(x_0) \|.$$

Taking $\tilde{l} = \frac{\nabla f(x_0)}{\| \nabla f(x_0) \|}$,

$$\frac{\partial f}{\partial \tilde{l}} = \langle \nabla f(x_0), \tilde{l} \rangle = \frac{1}{\| \nabla f(x_0) \|} \langle \nabla f(x_0), \nabla f(x_0) \rangle =$$

$$= \frac{\| \nabla f(x_0) \|^2}{\| \nabla f(x_0) \|} = \| \nabla f(x_0) \|.$$

$\blacksquare$

2. Derivatives of vector-valued functions

Let $D \subseteq \mathbb{R}^d$ and $f : D \to \mathbb{R}^m$.

Def 19.2 Let x_0 be an inner point of D. A function f is differentiable at x_0 if there exists a linear map

$$L : \mathbb{R}^d \to \mathbb{R}^m$$

such that

$$f(x) - f(x_0) - L(x - x_0) = o(\| x - x_0 \|), \quad x \to x_0.$$

Since each linear map can be defined by its matrix in the standard basis, namely,

$$L x = \begin{pmatrix} v_{11} & \dots & v_{1d} \\ - & - & - \\ v_{m1} & \dots & v_{md} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix},$$

we will identify L with its matrix

$$L = (v_{ij}).$$

The matrix $L = (v_{ij})$ is called the derivative of f at point x_0 and is denoted by $f'(x_0)$.

Th 19.4 A function $f : D \to \mathbb{R}^m$, $f = (f_1, \dots, f_m)$ is differentiable at x_0 iff $f_k : D \to \mathbb{R}$ is differentiable at x_0 for all $k = 1, \dots, m$.

Moreover
$$f'(x_0) = \left(\frac{\partial f_i}{\partial x_j}(x_0)\right)_{i=1,j=1}^{m,d} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \cdots & \frac{\partial f_1}{\partial x_d}(x_0) \\ - & - & - \\ \frac{\partial f_m(x_0)}{\partial x_1} & \cdots & \frac{\partial f_m(x_0)}{\partial x_d} \end{pmatrix}$$

Def 19.3 The matrix
$$f'(x_0) = \left(\frac{\partial f_i}{\partial x_j}(x_0)\right)_{i=1,j=1}^{m,d}$$
is called Jacobi matrix. If $m=d$, then the determinant
$$\frac{\partial(f_1,\dots,f_d)}{\partial(x_1,\dots,x_d)} = \det f'(x_0)$$
is called Jacobian. A point x_0 at which $\det f'(x_0)=0$ is called a singular point of f.

Th 19.5 (Chain rule II) Let D be an open set in $\mathbb{R}^d$ and M be an open set in $\mathbb{R}^m$. If $f: D \to M$ is differentiable at x_0 and $g: M \to \mathbb{R}^n$ is differentiable at $y_0 = f(x_0)$, then the function $h = g(f) = g\circ f : D \to \mathbb{R}^n$ is differentiable at x_0 and
$$h'(x_0) = g'(f(x_0))\, f'(x_0).$$

⑥ <u>Corollary 19.1</u> Under the assumptions of Th 19.5 and $d = m = n$

$$\frac{\partial(h_1, \dots, h_d)}{\partial(x_1, \dots, x_d)} = \frac{\partial(g_1, \dots, g_d)}{\partial(f_1, \dots, f_d)} \cdot \frac{\partial(f_1, \dots, f_d)}{\partial(x_1, \dots, x_d)}$$