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N 18 Differentiation of functions of several variables

1. Function of one variable.

Let $f : (a,b) \to \mathbb{R}$. The derivative of f at point x_0 is defined as

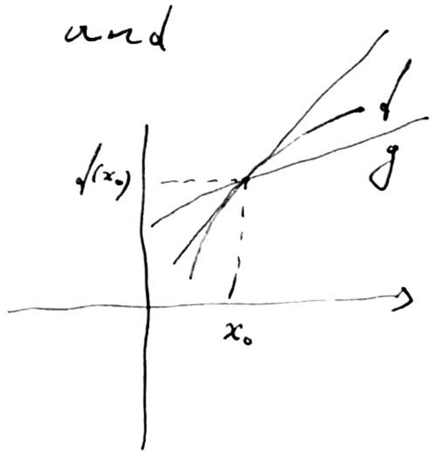
$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

if the limit exists.

We take any line

$$g(x) = f(x_0) + m(x - x_0)$$

through the point $(x_0, f(x_0))$ and consider the approximation



$$f(x) - g(x) = f(x) - f(x_0) - m(x - x_0)$$

Then

$$\frac{f(x) - g(x)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} - m \to 0, \quad x \to x_0$$

iff $m = f'(x_0)$.

Thus, $f(x) - g(x) = f(x) - f(x_0) - m(x - x_0) = o(x - x_0)$, $x \to x_0$

iff g is the tangent line to f at x_0, i.e. $m = f'(x_0)$.

2. Definition of differentiable functions

The function $L : \mathbb{R}^d \to \mathbb{R}$ is called linear if $L(x+y) = L(x) + L(y)$ and $L(ax) = a\,L(x)$ for all $x, y \in \mathbb{R}^d$, $a \in \mathbb{R}$.

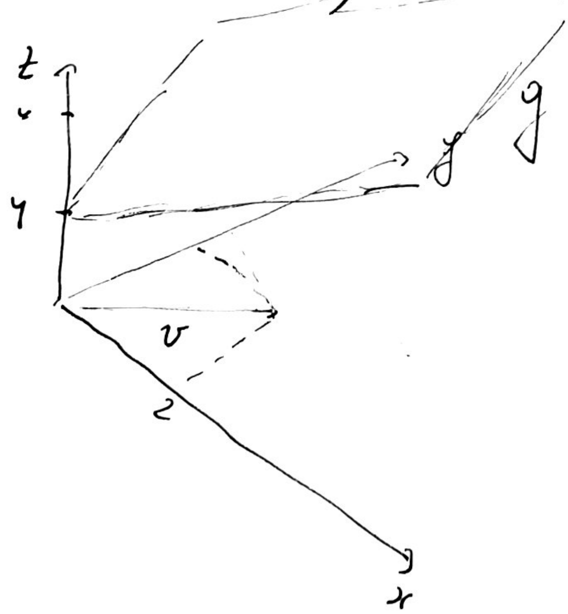
By the Riesz representation theorem (see Th 12.1), there exists a vector $v \in \mathbb{R}^d$, such that $\forall x \in \mathbb{R}^d$

$$L(x) = \langle v, x \rangle = v_1 x_1 + \dots + v_d x_d .$$

Similarly as before, we are going to approximate $f : \mathbb{R}^d \to \mathbb{R}$ by a function:

$$g(x) = a + L(x - x_0) = a + \langle v, x - x_0 \rangle$$

<u>Ex.18.1</u> Let $a = 4$, $v = (2, 2)$, $x_0 = (1, 0)$



$g(x)$ pass through (x_0, a) and the greatest increase of g is in the direction v.
Length of v corresponds for the rate of increase.

Def 18.1 Let $D \subseteq \mathbb{R}^d$ and x_0 be an inner point of D. Let also $f: D \to \mathbb{R}$.
The function f is called differentiable at x_0 if there exists a linear function $L(x) = \langle v, x \rangle$, $x \in \mathbb{R}^d$, such that

$$f(x) - f(x_0) - L(x - x_0) = o(\|x - x_0\|), \quad x \to x_0$$

that is

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - L(x - x_0)}{\|x - x_0\|} = 0.$$

The function

$$g(x) = f(x_0) + L(x - x_0), \quad x \in \mathbb{R}^d$$

is the tangent plane to f through the point $(x_0, f(x_0))$

Def 18.2 The function L is called the differential of f at x_0 and is denoted by

$$df(x_0) = L \quad \text{or} \quad df(x_0) = v_1 dx_1^0 + \ldots + v_d dx_d^0.$$

Ex 18.2 Take $d=2$, $f(x,y) = x^2+y^2$,

$$f(\underbrace{x_0+\Delta x}_{x}, \underbrace{y_0+\Delta y}_{y}) = x_0^2 + 2x_0\Delta x + (\Delta x)^2 + y_0^2 + 2y_0\Delta y + (\Delta y)^2 =$$
$$= f(x_0,y_0) + 2x_0\cdot\Delta x + 2y_0\cdot\Delta y + \|\Delta x\|^2$$

Then
$$\frac{f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) - L(\Delta x, \Delta y)}{\|\Delta x\|} = \|\Delta x\| \to 0 \quad \|\Delta x\| \to 0,$$

where $L(\Delta x, \Delta y) = \underbrace{2x_0}_{v_1}\cdot\Delta x + \underbrace{2y_0}_{v_2}\cdot\Delta y$

$$g(x,y) = \underbrace{x_0^2+y_0^2}_{f(x_0,y_0)} + \underbrace{2x_0}_{v_1}(x-x_0) + \underbrace{2y_0}_{v_2}(y-y_0)$$

- tangent plane at (x_0,y_0).

if $(x_0,y_0) = (\frac{1}{2}, \frac{1}{2})$, then

$$g(x,y) = x+y-\frac{1}{2}.$$

3. Partial derivatives

Let $e_1, \dots, e_d$ be a standard basis in $\mathbb{R}^d$

Def 18.3. The limit

$$\frac{\partial f}{\partial x_k}(x_0) = f'_{x_k}(x_0) = \lim_{t\to 0} \frac{f(x_0+te_k) - f(x_0)}{t} =$$

$$= \lim_{\Delta x_k \to 0} \frac{f(x_1^0, \dots, x_k^0 + \Delta x_k, \dots, x_d^0) - f(x_1^0, \dots, x_k^0, \dots, x_d^0)}{\Delta x_k},$$

if it exists, is called the partial derivative of f with respect to x_k.

<u>Def 18.4</u> The vector

$$\operatorname{grad} f(x_0) = \nabla f(x_0) = \left(\frac{\partial f}{\partial x_1}(x_0), \dots, \frac{\partial f}{\partial x_d}(x_0) \right)$$

is called the gradient of f

<u>Ex 18.3</u> $f(x,y) = x^2 + y^2$

$$\frac{\partial f}{\partial x} = 2x + 0 = 2x$$

$$\frac{\partial f}{\partial y} = 0 + 2y = 2y.$$

<u>Exercise 18.1</u> For functions

a) $f(x_1, \dots, x_d) = \sum_{i=1}^{d} a_i x_i$

б) $f(x_1, \dots, x_d) = \sum_{i,j=1}^{d} a_{ij} x_i x_j$

compute ∇f.

<u>Exercise 18.2</u> Show that

1) $\nabla(cf) = c\nabla f$ for c constant

2) $\nabla(f+g) = \nabla f + \nabla g$

3) $\nabla(fg) = g\nabla f + f\nabla g$

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4) $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$, $g \neq 0$.

Th 18.1 Let f be differentiable at x_0. Then for every $k=1,\dots,d$ there exists a partial derivative of f with respect to x_k and the differential of f is defined as

$$df(x_0) = \frac{\partial f}{\partial x_1}(x_0)dx_1 + \dots + \frac{\partial f}{\partial x_d}(x_0)dx_d$$

that is, the linear map in Def 18.1 has a form

$$Lx = \langle \nabla f(x_0), x \rangle$$

Remark 18.1 $g(x) = f(x_0) + \nabla f(x_0)(x - x_0)$

is the tangent plane to the graph of g through the point $(x_0, f(x_0))$.

Proof of Th 18.1 We compute

$$0 = \lim_{t \to 0} \frac{f(x_0 + te_k) - f(x_0) - L(te_k)}{\|te_k\|} =$$

$$= \lim_{t \to 0} \frac{f(x_0 + te_k) - f(x_0)}{t \underbrace{\|e_k\|}_{=1}} - \underbrace{Le_k}_{=v_k}$$

$$= \frac{\partial f}{\partial x_k}(x_0)$$

∎

Th 18.2 If f is differentiable at x_0, then f is continuous at x_0.

Proof Consider

$$f(x) - f(x_0) = \underbrace{L(x-x_0)}_{\downarrow\, 0} + \underbrace{o(\|x-x_0\|)}_{\downarrow\, 0}$$

as $x \to x_0$, since L is continuous.
Thus, $f(x) \to f(x_0)$, $x \to x_0$. ∎

Exercise 18.3 For which $\alpha \geq 0$ is the function

$$f(x_1, \dots, x_d) = \left(\sum_{k=1}^{d} x_k^2\right)^{\frac{\alpha}{2}} = \|x\|^{\alpha}$$

differentiable at $x_0 = (0, \dots, 0)$.

Th 18.3 Let x_0 be an inner point of D and $f: D \to \mathbb{R}$. If

1) $\exists\ \varepsilon > 0$, $\forall z \in B_\varepsilon(x_0)$ $\exists\ \frac{\partial f}{\partial x_k}(z)$ $\forall k$;
2) $\frac{\partial f}{\partial x_k}$ is continuous at x_0 $\forall k$,

then f is differentiable at x_0

Exercise 18.3 Show that $f(x,y) = \frac{xy}{x^2+y^2}$ has partial derivatives at each $(x_0, y_0) \in \mathbb{R}^2$, but f is discont. at $(0,0)$ and consequently not differentiable.

Corollary 18.1 If D is open set and $f: D \to \mathbb{R}$ has continuous partial derivatives on D then f is differentiable at each point of D

$C^1(D)$ will denote the set of all differentiable functions on D.

Thus

$$f \in C^1(D) \iff \forall k \ \frac{\partial f}{\partial x_k} \in C(A).$$

Th 18.4 If $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ are differentiable at $x_0 \in D$. Then cf, $f+g$, fg, $\frac{f}{g}$ (if $g(x_0) \neq 0$) are differentiable at x_0

Th 18.5 (Chain rule) Let $f: D \to \mathbb{R}$ be differentiable at x°. Let also

$x_k = x_k(t_1, \dots, t_m)$, such that

$x_k^\circ = x_k(t_1^\circ, \dots, t_m^\circ)$ and $\exists \ \frac{\partial x_k}{\partial t_j}(t^\circ) \ \forall k, j$

Then for the function $h(t) = f(x(t))$ there exists a partial derivatives

$$\frac{\partial h}{\partial t_j}(t_0) = \sum_{k=1}^{d} \frac{\partial f}{\partial x_k}(x_0) \frac{\partial x_k}{\partial t_j}(t_0).$$