N 17. Continuous functions

1. Definition and basic properties.

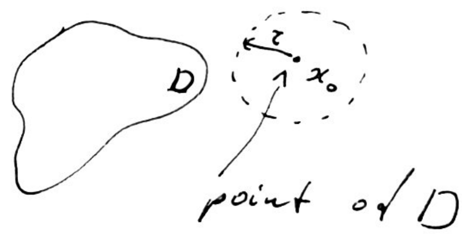
Let $D \subseteq \mathbb{R}^d$ and $f: D \to \mathbb{R}^m$.

Def 17.1 Let $x_0 \in D$ be a limit point of D. A function $f: D \to \mathbb{R}^m$ is called continuous at the point x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0),$$

i.e. $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in D, \ \|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \varepsilon$.

Def 17.2 A point $x_0 \in D$ is called an isolated point of D if $\exists r > 0$: $B_r(x_0) \cap D = \{x_0\}$



We assume that any function is continuous at an isolated point.

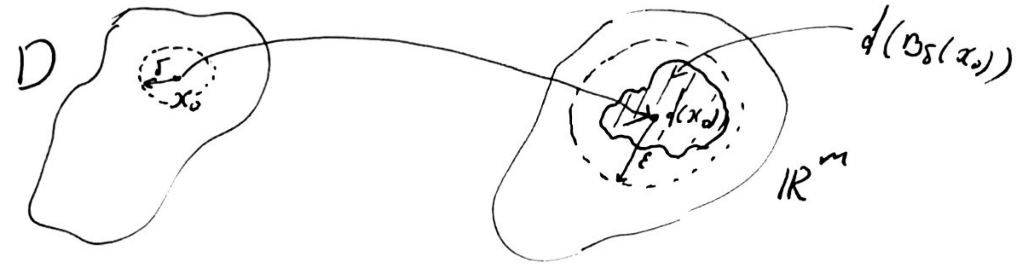
Def 17.3 A function $f: D \to \mathbb{R}^m$ is continuous on D if f is continuous at each point of D.

Notation: $f \in C(D, \mathbb{R}^m)$.

If $m = 1$, we denote $C(D, \mathbb{R}) =: C(D)$.

<u>Remark 17.1</u> If x_0 is an inner point of D then f is continuous at x_0 iff

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{s.t.} \quad f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$$



<u>Th. 17.1</u> Let $f: D \to \mathbb{R}$, $g: D \to \mathbb{R}$ be continuous at $x_0 \in D$. Then

a) $\forall c \in \mathbb{R}$ cf is cont. at x_0

b) $f+g$ is cont. at x_0

c) fg is cont. at x_0

d) $\frac{f}{g}$ is cont. at x_0, if $g(x_0) \neq 0$.

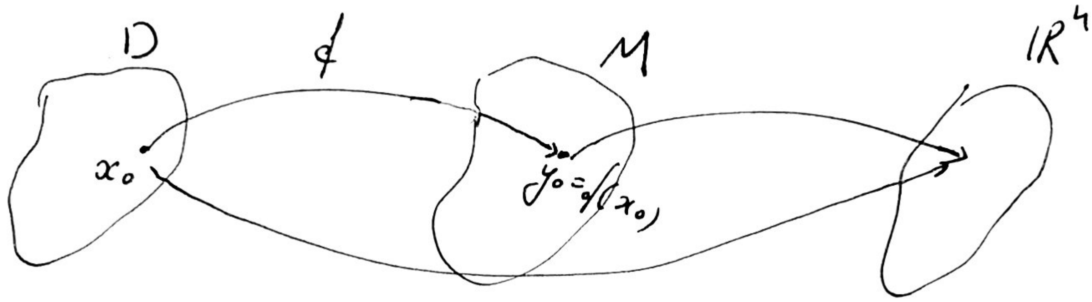
<u>Th 17.2</u> Let $f = (f_1, \dots, f_m): D \to \mathbb{R}^m$.
f is continuous at x_0 (as a map from D to $\mathbb{R}^m$) iff $\forall i = 1, \dots, m$ f_i is continuous at x_0 (as a map from D to $\mathbb{R}$).

<u>Ex. 17.1</u> The function

$$f(t) = (\cos t, \sin t), \quad t \in \mathbb{R},$$

is a comtinuous function from $\mathbb{R}$ to $\mathbb{R}^2$ since $f_1(t) = \cos t$ and $f_2(t) = \sin t$ are continuous.

Th 17.3 (Composition of continuous functions)
Let $f: D \to M$, where $M \subseteq \mathbb{R}^m$ and $g: M \to \mathbb{R}^n$. Let also f be continuous at $x_0 \in D$ and g be continuous at $y_0 = f(x_0)$. Then the function $h(x) = g(f(x)) = (g \circ f)(x)$ is continuous at x_0



2. Examples of continuous functions

1) The constant function

$$f(x) = c, \quad x \in \mathbb{R}^d,$$

where c is some constant from $\mathbb{R}$

2) Coordinate functions

$$\pi_k : \mathbb{R}^d \to \mathbb{R}, \quad k = 1, \dots, d$$

$$\pi_k(x_1, \dots, x_d) = x_k, \quad x \in \mathbb{R}^d,$$

The continuity follows from

$$|x_k - y_k| = |\pi_k(x) - \pi_k(y)|$$

$$\| $$

$$\sqrt{(x_k - y_k)^2} \leq \sqrt{(x_1 - y_1)^2 + \dots + (x_d - y_d)^2} = \|x - y\|$$

So, if $\|x - x_0\| \to 0$, then $|\pi_k(x) - \pi_k(x_0)| \to 0$.

3) Polynomials.

$$P(x_1,\dots,x_d)=\sum_{k_1=0}^{n_1}\dots\sum_{k_d=0}^{n_d} a_{k_1\dots k_d}\, x_1^{k_1}\dots x_d^{k_d}\quad,\ x\in\mathbb{R}^d.$$

The continuity follows from Th 17.1.

4) Rational functions

Let P and Q be polynomials and
$D=\{x\in\mathbb{R}^d:\ Q(x)\neq 0\}$.

The function $\frac{P}{Q}$ is continuous on D.

5) Other functions

$$f(x_1,x_2)=e^{\sqrt{x_1^2+x_2^2}}\sin(x_1+x_1x_2^2)\quad,(x_1,x_2)\in\mathbb{R}^2.$$

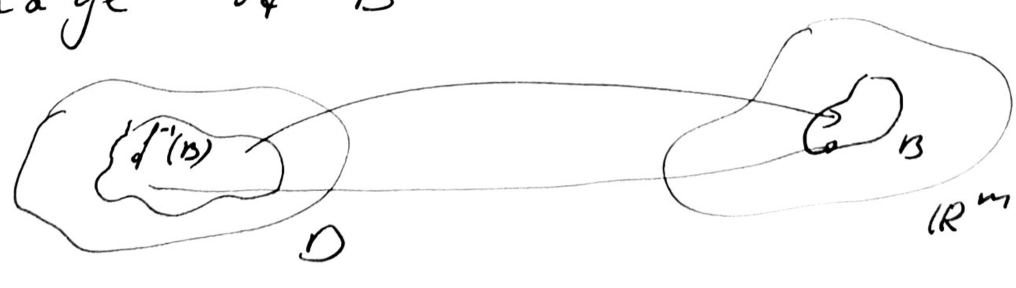
The continuity follows from the continuity of exponent function, sin, $\sqrt{\ }$ and Th 17.3.

3. Characterization of continuous functions.

Let $f: D\to\mathbb{R}^m$, and $B\subseteq\mathbb{R}^m$. We set

$$f^{-1}(B)=\{x\in D:\ f(x)\in B\}$$

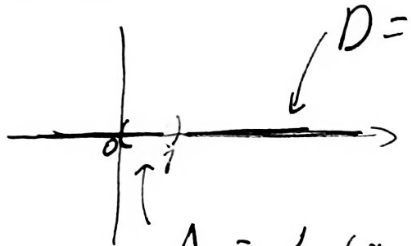
– preimage of B



If $A \subseteq D$, then we say that A is **open in D** if $\forall x \in A \ \exists \tau > 0$:

$$B_\tau(x) \cap D \subseteq A$$

Ex 17.2



$D = \{(x,y) : y = 0\}$

$A = \{(x,y) : y = 0, 0 < x < 1\}$ – open in D but is not open in $\mathbb{R}^2$.

We remark that if D is open, then A is open in D iff A is open in $\mathbb{R}^d$.

Th 17.4 Let $f : D \to \mathbb{R}^m$. The function f is continuous on D iff $\forall G$ - open in $\mathbb{R}^m$ $f^{-1}(G)$ is open in D.

Proof $\Rightarrow$) Let f be continuous.

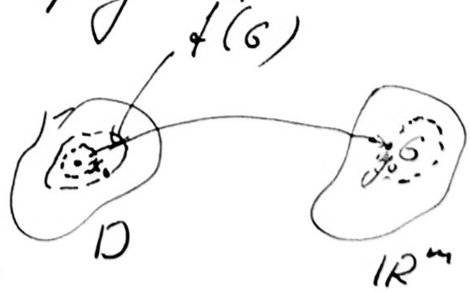
Let G be open and non-empty.

Take $x_0 \in f^{-1}(G)$.

Set $y_0 := f(x_0) \in G$.

Since G is open, $\exists \varepsilon > 0$ s.t.

$$B_\varepsilon(y_0) \subseteq G$$



By the continuity of f at x_0, we have that $\exists\ \delta > 0$ s.t.

$$\forall x \in B_\delta(x_0) \cap D \qquad f(x) \in B_\varepsilon(y_0) \subseteq G$$

So, $B_\delta(x_0) \cap D \subseteq f^{-1}(G)$.

Thus $f^{-1}(G)$ is open.

$\Leftarrow$) Let $x_0 \in D$ and $\varepsilon > 0$. The set $G = B_\varepsilon(y_0)$ is open in $\mathbb{R}^m$, where $y_0 := f(x_0)$.
Thus, $f^{-1}(B_\varepsilon(y_0))$ is open in D.
It implies, that $\exists\ \delta > 0$ s.t.

$$B_\delta(x_0) \cap D \subseteq f^{-1}(B_\varepsilon(y_0))$$

$$\Rightarrow \quad \|x - x_0\| < \delta,\ x \in D \text{ then}$$

$$\|f(x) - \underset{=f(x_0)}{y_0}\| < \varepsilon$$

4. Continuous functions on compact sets

We recall that K is compact in $\mathbb{R}^d$ if any open cover of K contains a finite subcover of K.

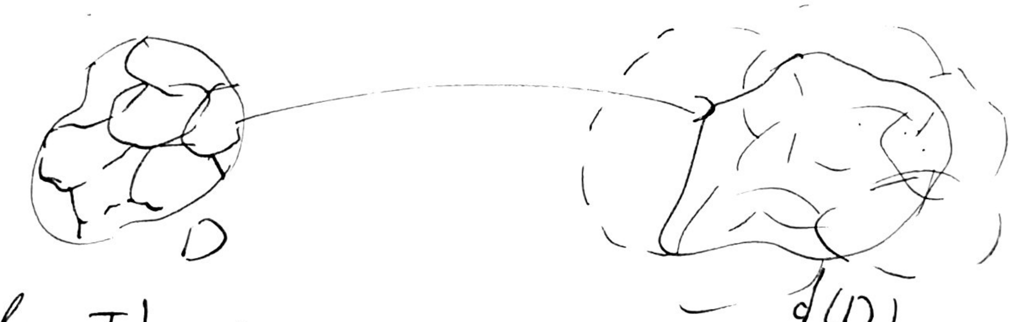
7 <u>Exercise 17.1</u> Show that K is compact iff $\forall\, G_\alpha,\ \alpha \in T$, open in K s.t.

$$\bigcup_{\alpha \in T} G_\alpha \supseteq T$$

there exists $\alpha_1, \dots, \alpha_n$ s.t. $T \subseteq \bigcup_{i=1}^n G_{\alpha_i}$.

<u>Th. 17.5</u> Let D be a compact set in $\mathbb{R}^d$ and $f \in C(D, \mathbb{R}^m)$. Then $f(D)$ compact in $\mathbb{R}^m$.

<u>Proof.</u> Let $G_\alpha, \alpha \in T$, be an open cover of $f(D)$



Then by Th. 17.4, the sets $f^{-1}(G_\alpha)$, $\alpha \in T$, are open in D, moreover, they cover D. Since D is compact, Exercise 17.1 implies, that $\exists\ \alpha_1, \dots, \alpha_n$ s.t.

$$D \subset \bigcup_{i=1}^n f^{-1}(G_\alpha)$$

Hence

$$f(D) \subset f\Big(\bigcup_{i=1}^n f^{-1}(G_{\alpha_i})\Big) \subseteq \bigcup_{i=1}^n G_{\alpha_i}.$$

Thus, $f(D)$ is compact.

8 **Th 17.6** Let K be a compact and $f: K \to \mathbb{R}$ be continuous.

Then 1) f is bounded on K i.e. $\exists\, C > 0$:

$$|f(x)| \le C \quad \forall\, x \in K$$

2) $\exists\, x_*, x^* \in K$ such that

$$f(x_*) = \min_{x \in K} f(x), \quad f(x^*) = \max_{x \in K} f(x).$$

__Proof.__ Since K is compact, $f(K) = \{f(x) : x \in K\}$ is compact in $\mathbb{R}$. Hence $f(K)$ is bounded that implies 1).

Moreover, $f(K)$ is closed, thus it contains its supremum and infimum. Let $y^* = \sup f(K) \in f(K)$. Then, $\exists\, x^*$ s.t. $f(x^*) = y^*$.

The same for the infimum.

∎

Th 17.7. Let K be a compact and $f: K \to \mathbb{R}^m$ be continuous. Then f is uniformly continuous, that is,

$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0 \;\; \forall\, x', x'' \in K$

$$\|x' - x''\| < \delta \Rightarrow \|f(x') - f(x'')\| < \varepsilon.$$