N 16 Functions of several variables

1. Compact sets in $\mathbb{R}^d$

We recall that

- A is open if each point of A is its inner point i.e.

$$\forall x \in A \quad \exists \varepsilon > 0 \quad : B_\varepsilon(x) = \{y \in \mathbb{R}^d : \|x-y\| < \varepsilon\} \subseteq A$$

- A is closed if it contains all its limit points

(x_0 is a limit point of A if $\forall \varepsilon > 0 \ \exists x \neq x_0$, $x \in A \cap B_\varepsilon(x_0)$

$\Longleftrightarrow \exists \ (x^{(n)})_{n \geq 1}$: 1) $x^{(n)} \neq x_0$, $x^{(n)} \in A$

2) $x^{(n)} \to x_0$)

A is closed $\Longleftrightarrow$ $\mathbb{R}^d \setminus A$ is open

Def 16.1 By an open cover of $A \subseteq \mathbb{R}^d$ we mean a collection G_α, $\alpha \in T$, of open subsets of $\mathbb{R}^d$ such that

$$A \subseteq \bigcup_{\alpha \in T} G_\alpha$$

Def 16.2 A subset $K \subseteq \mathbb{R}^d$ is said to be compact if every open cover G_α, $\alpha \in T$, of K contains a finite subcover, i.e.

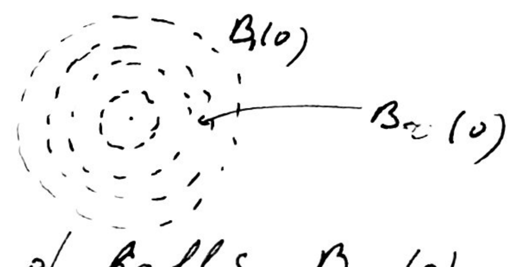
$$\exists \ \alpha_1, \dots, \alpha_n \text{ such that } K \subseteq G_{\alpha_1} \cup \dots \cup G_{\alpha_n}.$$

<u>Ex 16.1</u> The open ball $B_1(0) = \{ y \in \mathbb{R}^d : \|y\| < 1 \}$ is not compact

(The same for any $B_r(x)$)

Indeed, take $G_r = B_r(0) = \{ y \in \mathbb{R}^d : \|y\| < r \}$

Then $B_1(0) \subseteq \bigcup_{0<r<1} G_r$



But $B_1(0)$ can not be covered by a finite number of balls $B_r(0)$, $r<1$.

<u>Th 16.1</u> Let $K \subseteq \mathbb{R}^d$. The following statements are equivalent:

1) K is compact;

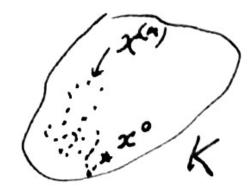
2) K is bounded and close;

3) any sequence $(x^{(n)})_{n \geq 1}$ of K has a convergent subsequence i.e.

$\forall \, (x^{(n)})_{n \geq 1}$, $x^{(n)} \in K$, one has

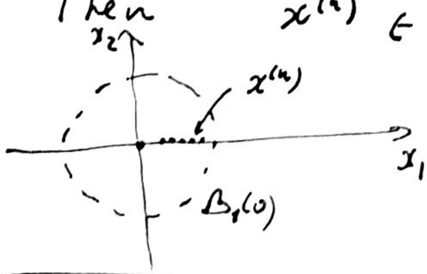
$\exists \, n_1, n_2, \dots$ such that $x^{n_k} \to x_0$, $k \to \infty$ and $x_0 \in K$

<u>Remark 16.1</u> If we take $d=2$,



$K = B_1(0)$ and $x^{(n)} = (1-\frac{1}{n}, 0)$

Then $x^{(n)} \in B_1(0)$ but $x^{(n)} \to (1,0) \notin B_1(0)$.



2. Examples of functions of several variables

a) $f : \overset{\subseteq \mathbb{R}}{D} \to \mathbb{R}$ – real-valued function of one variable

$f(x) = 2x \qquad x \in \mathbb{R}$

$f(x) = \sin x, \quad x \in \mathbb{R}$

$f(x) = \sqrt{1-x^2}, \quad x \in [-1,1]$

b) $f : \overset{\subseteq \mathbb{R}^d}{D} \to \mathbb{R}$ – real-valued multivariate function of functions of several variables

$f(x_1,x_2) = 3x_1 + 2x_2, \quad (x_1, x_2) \in \mathbb{R}^2$

$f(x_1,x_2) = x_1^2 + x_2^2, \quad (x_1, x_2) \in \mathbb{R}^2$

$z = f(x, y) = \sin x \sin y, \quad (x,y) \in \mathbb{R}^2$

The set $\{x \in D : f(x) = a\} =: D_a$

is called a level set of f.

c) $f : \overset{\subseteq \mathbb{R}^d}{D} \to \mathbb{R}^m$ – vector-valued functions

$f(x_1, \dots, x_d) = (f_1(x_1,\dots,x_d), \dots, f_m(x_1,\dots,x_d))$

For instance, such functions can associate to each point of fluid its velocity vector

:

$f(x,y) = (\cos x \sin y, \ \sin x \cos y) \ \left(= \nabla \sin x \sin y\right)$

$f(x_0, y_0)$ – the direction of gratest increase of the function $g(x,y) = \sin x \sin y$ at (x_0, y_0) and its magnitude is the rate of increase in that direction.

d) $f : D^{\subseteq \mathbb{R}} \to \mathbb{R}^m$ – vector-valued function of single variable

$$f(t) = (1+2t, \ t, \ 3-t) \ , \quad t \in \mathbb{R}$$

$$f(t) = (\cos t, \ \sin t) \in \mathbb{R}^2 \ , \quad t \in [0, 2\pi)$$

$$f(t) = (\cos t, \ \sin t, \ \tfrac{t}{2\pi}) \in \mathbb{R}^3 \ , \quad t \in [0, 2\pi)$$

3 Limit of function

Def 16.3 Let x_0 be a limit point of $D \subseteq \mathbb{R}^d$ and $f : D \to \mathbb{R}^m$. The point $p \in \mathbb{R}^m$ is the limit of f at point x_0 if for every sequence $(x^{(n)})_{n \geq 1}$ such that $x^{(n)} \in D$, $x^{(n)} \neq x_0$, $x^{(n)} \to x_0$ one has $f(x^{(n)}) \to p$, $n \to \infty$.

Notation: $p = \lim\limits_{x \to x_0} f(x)$

Th 16.2 Let x_0 be a limit point of D and $f: D \to \mathbb{R}^m$. Then $p = \lim_{x \to x_0} f(x)$ iff $\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D,\ x \neq x_0,\ \|x - x_0\| < \delta$ one has $\|f(x) - p\| < \varepsilon$.

Remark 16.2. If $f: D \to \mathbb{R}^m$ and $f = (f_1, \dots, f_m)$. Then $p = \lim_{x \to x_0} f(x)$ iff $p_i = \lim_{x \to x_0} f_i(x), \forall i$, where $p = (p_1, \dots, p_m)$

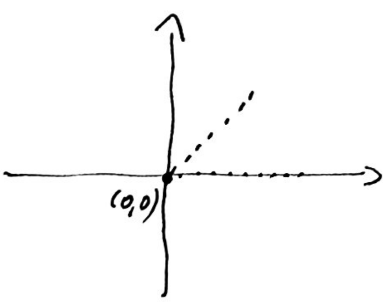
Ex 16.2 a) We show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0$$

$$0 \le \left| \frac{x^2 y}{x^2+y^2} \right| = \frac{x^2 |y|}{x^2+y^2} \le \frac{(x^2+y^2)|y|}{x^2+y^2} = |y| \to 0$$

$\rightsquigarrow 0$ by the squeeze theorem.

b) $\lim_{(x,y)\to(0,0)} \underbrace{\frac{xy}{x^2+y^2}}_{= f(x,y)}$ does not exists



Consider $(x_n, y_n) := (x_n, -x_n)$, $x_n \to 0$

$$f(x_n, y_n) = \frac{-x_n^2}{x_n^2 + x_n^2} = -\frac{1}{2} \to \frac{1}{2}, \quad n \to \infty$$

Consider $(x_n, y_n) := (x_n, x_n)$, $x_n \to 0$

$$f(x_n, x_n) = \frac{x_n^2}{x_n^2 + x_n^2} = \frac{1}{2} \to \frac{1}{2}, \quad n \to \infty$$

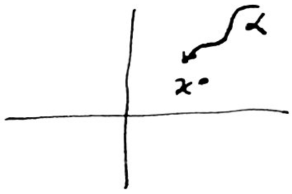
Thus, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exists.

⑥

Th 16.3 Let $f : \underset{\mathbb{R}^d}{\underset{\cap}{D}} \to \mathbb{R}^m$ and x_0 be a limit point of D. Then $\lim_{x \to x_0} f(x) = p$ iff for any map $\alpha : (0, \varepsilon) \to D$ such that 1) $\lim_{t \to 0} \alpha(t) = x_0$

2) $\alpha(t) \neq x_0 \quad \forall t \in (0, \varepsilon)$

one has $\lim_{t \to 0} f(\alpha(t)) = p.$



We remark that in Th 16.3 it is not enough to take α to be only lines

$$\alpha(t) = x_0 + at \quad , t > 0.$$

Ex 16.3 The limit $\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^4 + y^2}$ exists along every line but in general it does not exists. Indeed, let

$$\begin{cases} x = at \\ y = bt \end{cases} , \quad t > 0$$

Then

$$\frac{x^2 y}{x^4 + y^2} = \frac{a^2 t^2 bt}{a^4 t^4 + b^2 t^2} = \frac{a^2 b t}{a^4 t^2 + b^2} \to 0, \quad t \to 0$$

But for $\begin{cases} x = t \\ y = t^2 \end{cases} , \quad t > 0$

$$\frac{x^2 y}{x^4 + y^2} = \frac{t^2 t^2}{t^4 + t^4} = \frac{1}{2} \to \frac{1}{2}, \quad t \to 0. \text{ Thus,}$$

$\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^4 + y^2}$ does not exists.

⑦

<u>Remark 16.3</u> If $f(x) = g(\|x\|)$, then

$$\lim_{x\to 0} f(x) = \lim_{\|x\|\to 0} g(\|x\|) = \lim_{r\to 0} g(r).$$

<u>Ex 16.4</u> $\lim_{x\to 0} (x^2+y^2)\ln(x^2+y^2) = \lim_{r\to 0} r^2 \ln r = 0,$

since $(x^2+y^2)\ln(x^2+y^2) = \|(x,y)\|^2 \ln \|(x,y)\|,$

where $\|(x,y)\| = \sqrt{x^2+y^2}.$