Lecture N 15 Topology in $\mathbb{R}^d$

1. Norm in $\mathbb{R}^d$

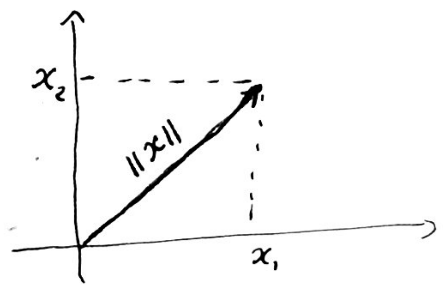
We recall that the inner product on $\mathbb{R}^d$ is defined as

$$\langle x, y \rangle = x_1 y_1 + \dots + x_d y_d$$

for $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$.

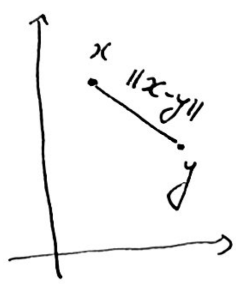
The norm on $\mathbb{R}^d$ is defined as

$$\| x \| = \sqrt{\langle x, x \rangle} = \sqrt{x_1^2 + \dots + x_d^2}.$$



We define the distance between x and y from $\mathbb{R}^n$ as the number

$$\| x - y \| = \sqrt{(x_1 - y_1)^2 + \dots + (x_d - y_d)^2}$$



Properties of norm and distance:

1) $\| x \| \geq 0 \quad \forall x \in \mathbb{R}^d$

2) $\| \alpha x \| = |\alpha| \, \| x \| \quad \forall x \in \mathbb{R}^d, \ \alpha \in \mathbb{R}$

For norm

3) $\| x \| = 0 \iff x = (0, \dots, 0)$

4) $\| x + y \| \leq \| x \| + \| y \|$ (triangle inequality)

5) $| \, \| x \| - \| y \| \, | \leq \| x - y \|$

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6) $\|x-y\| \geq 0 \quad \forall x, y \in \mathbb{R}^d$

7) $\|x-y\| = \|y-x\|, \quad \forall x, y \in \mathbb{R}^d$

8) $\|x-y\| = 0 \iff x = y$

9) $\|x-y\| \leq \|x-z\| + \|z-y\| \quad \forall x, y, z \in \mathbb{R}^d$
(triangle inequality)

10) $\big| \|x-z\| - \|y-z\| \big| \leq \|x-y\|$

For distance

2. Limits in $\mathbb{R}^d$

Let $(x^{(n)})_{n\geq 1}$ be a sequence of elements from $\mathbb{R}^d$.

Def 15.1 A sequence $(x^{(n)})_{n\geq 1}$ converges in $\mathbb{R}^d$ if there exists $x \in \mathbb{R}^d$ such that

$$\|x_n - x\| \to 0, \quad n \to \infty,$$

i.e. $\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad \|x_n - x\| < \varepsilon$.

Th 15.1 If $x^{(n)} \to x$ and $x^{(n)} \to y$, $n \to \infty$, then $x = y$.

Proof $\quad 0 \leq \|x-y\| \overset{9)}{\leq} \|x - x_n\| + \|x_n - y\| \underset{n\to\infty}{\longrightarrow} 0.$

Hence, $\|x-y\| = 0 \overset{8)}{\Longrightarrow} x = y$.

<u>Th 15.2</u> Let $x^{(n)} \to x$, $n \to \infty$. Then for every $y \in \mathbb{R}^d$ $\|x^{(n)} - y\| \to \|x - y\|$, $n \to \infty$.

<u>Proof.</u>

$$0 \leq \left| \|x^{(n)} - y\| - \|x - y\| \right| \overset{10)}{\leq} \|x^{(n)} - x\| \to 0$$

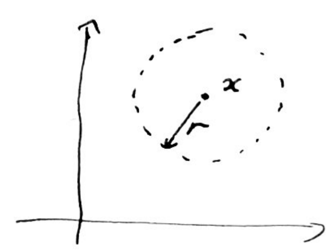
Thus, $\|x^{(n)} - y\| \to \|x - y\|$.

3. Limit points in $\mathbb{R}^d$.

<u>Def 15.2</u> The set

$$B_r(x) = \{ y \in \mathbb{R}^d : \|x - y\| < r \}$$

is called <u>an open ball</u> of radius $r > 0$ and center $x \in \mathbb{R}^d$.



<u>Def 15.3</u> The set

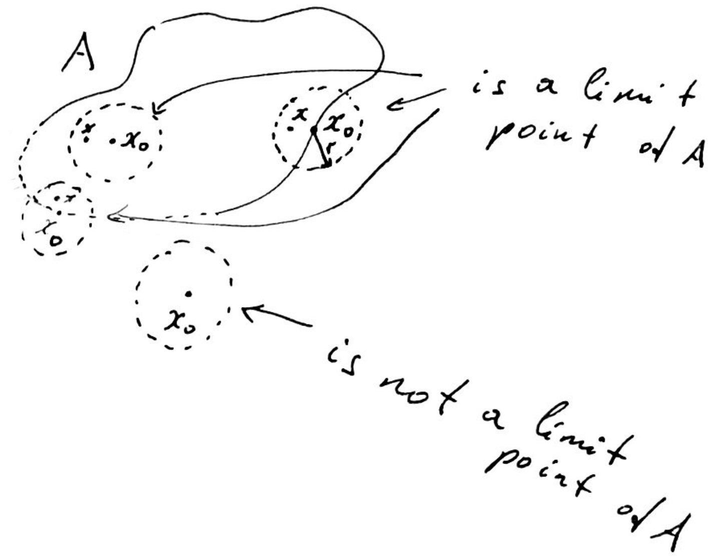
$$\overline{B_r}(x) = \{ y \in \mathbb{R}^d : \|x - y\| \leq r \}$$

is called a <u>closed ball</u> of radius $r > 0$ and center $x \in \mathbb{R}^d$.

<u>Def 15.4</u> A set $A \subseteq \mathbb{R}^d$ is bounded if there exists $r > 0$ such that

$$A \subseteq B_r(0).$$

__Def 15.5__ Let $A \subseteq \mathbb{R}^d$ and $x_0 \in \mathbb{R}^d$. The point x_0 is a limit point of A if for every $r > 0$ $\exists\, x \in A$, $x \neq x_0$: $x \in B_r(x_0)$



__Th 15.3__ A point $x_0 \in \mathbb{R}^d$ is a limit point of $A \subseteq \mathbb{R}^d$ iff there exists $(x^{(n)})_{n \geq 1}$ such that

1) $\forall n$ $x^{(n)} \in A$, $x^{(n)} \neq x_0$

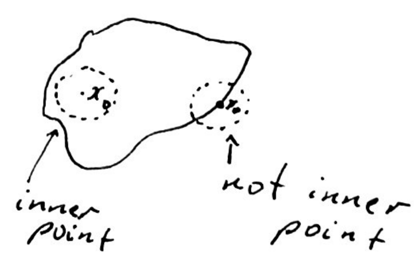
2) $x^{(n)} \to x_0$, $n \to \infty$.

__Exercise 15.1__ Prove Th. 15.3.

4. Open sets

__Def 15.5__ A point $x_0 \in A$ is called an inner point of A if $\exists\, r > 0$: $B_r(x_0) \subseteq A$

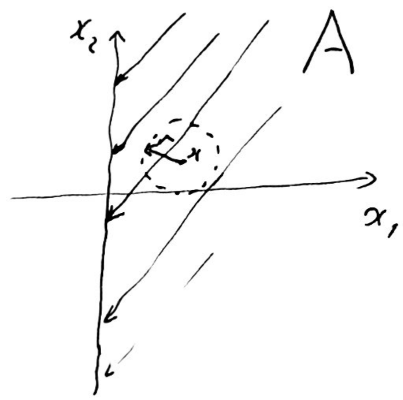
__Def 15.6__ A set $A \subseteq \mathbb{R}^d$ is called __open__ if each point of A is its inner point, i.e.



$\forall\, x \in A$ $\exists\, r > 0$: $B_r(x) \subseteq A$.

<u>Exercise 15.2.</u> Show that an open ball is an open set (according to Def 15.6).

<u>Ex. 15.1</u> The set $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$ is open



If $x = (x_1, x_2) \in A$, we can take $r = x_1 > 0$. Then $B_r(x) \subseteq A$. Indeed if $y \in B_r(x) \Rightarrow \|x - y\| < r \Rightarrow (x_1 - y_1)^2 + \underbrace{(x_2 - y_2)^2}_{\geq 0} < \overset{x_1}{r^2}$

$\Rightarrow (x_1 - y_1)^2 < x_1^2 \Rightarrow |y_1 - x_1| < x_1 \Rightarrow -x_1 < y_1 - x_1 < x_1 \Rightarrow$

$\Rightarrow 0 < y_1 < 2x_1 \Rightarrow y \in A$ (since $y_1 > 0$).

<u>Exercise 15.3</u> Show that

$$A_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < x_2\}$$

$$A_2 = \{x \in \mathbb{R}^d : x_1 > 0\}$$

$$A_3 = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 < 0\}$$

are open.

<u>Remark 15.1</u> The set $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\}$ is not open, since points $(0, x_2)$, $x_2 \in \mathbb{R}$, are not inner points of A.

⑥

Th 15.4 A union of any number of open sets is open.

Proof. Let A_α, $\alpha \in T$, be open sets. We prove that the set

$$\bigcup_{\alpha \in T} A_\alpha = \{ x : \exists \alpha \in T, \ x \in A_\alpha \}$$

is open.

Take any point $x \in \bigcup_{\alpha \in T} A_\alpha$, then $\exists \alpha \in T$ such that $x \in A_\alpha$. Since A_α is open, x is its inner point. So, $\exists r > 0$ such that $B_r(x) \subseteq A_\alpha \Rightarrow B_r(x) \subseteq \bigcup_{\alpha \in T} A_\alpha$.

We have obtained that x is inner point of $\bigcup_{\alpha \in T} A_\alpha$. This implies that $\bigcup_{\alpha \in T} A_\alpha$ is open. ■

Th 15.5 Intersection of a finite number of open sets is open.

Proof. Let $x \in \bigcap_{k=1}^{n} A_k$. Then $x \in A_k \ \forall k = 1, \dots, n$.

open

$$\boxed{\bigcap_{k=1}^{n} A_k = \{ x : x \in A_k \ \forall k = 1, \dots n \}}$$

So, $\exists r_k > 0 : B_{r_k}(x) \subseteq A_k \ \forall k$

Take $r = \min\{ r_1, \dots, r_k \}$

It is clear that $B_r(x) \subseteq B_{r_k}(x) \subseteq A_k \ \forall k$.

Thus, $B_r(x) \subseteq \bigcap_{k=1}^{n} A_k$. ■

Remark. 15.2 Intersection of any number of open sets is not open in general.

Ex. 15.2 $A = \bigcap_{r>0} \underset{\substack{\uparrow \\ \text{open}}}{B_r(x)} = \{x\}$ - is not open

5 Closed sets

Def 15.7 A set A is closed if it contains all its limit points.

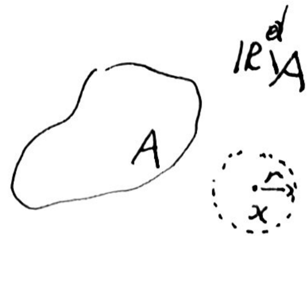
Th 15.6 A set $A \subseteq \mathbb{R}^d$ is closed iff

$$\mathbb{R}^d \setminus A = \{x \in \mathbb{R}^d : x \notin A\}$$

is open.

Proof $\Rightarrow$) let A be closed. Take $x \in \mathbb{R}^d \setminus A \Leftrightarrow x \notin A$. Thus, x is not a limit point of A and hence $\exists r>0$: $B_r(x) \cap A = \emptyset \Rightarrow B_r(x) \subseteq \mathbb{R}^d \setminus A$. So, $\mathbb{R}^d \setminus A$ is open.

$\Leftarrow$) Let $\mathbb{R}^d \setminus A$ is open. Then any $x \in \mathbb{R}^d \setminus A$ is an inner point of $\mathbb{R}^d \setminus A$. So, $\exists r>0$: $B_r(x) \subseteq \mathbb{R}^d \setminus A \Rightarrow B_r(x) \cap A = \emptyset$. Thus x is not a limit point of A.



Th 15.7 • Intersection of any number of close sets is a closed set.

• Union of a finite number of close sets is a closed set.

Proof Follows from the fact

$$\mathbb{R}^d \setminus \left(\bigcup_{\alpha \in T} A \right) = \bigcap_{\alpha \in T} (\mathbb{R}^d \setminus A)$$

$$\mathbb{R}^d \setminus \left(\bigcap_{k=1}^{n} A_k \right) = \bigcup_{k=1}^{n} (\mathbb{R}^d \setminus A_k),$$

and theorems 15.4, 15.5, 15.6.

Recall: $\bigcap_{\alpha \in T} A_\alpha = \{ x : x \in A_\alpha \ \forall \alpha \in T \}$.

Def 15.8 The set $\bar{A}$ that consists of all points of A and all limit points of A is called the closure of A.

Ex. 15.3 $A = \{ x \in \mathbb{R}^2 : x_1 > 0 \}$

$\bar{A} = \{ x \in \mathbb{R}^2 : x_1 \geq 0 \}$

$\overline{B_r(x)} = \{ y : \|x - y\| \leq r \} = \bar{B}_r(x)$.

Exercise 15.4 a) Show that $\bar{A}$ is a closed set

b) Show that $\bar{\bar{A}} = \bar{A}$

c) Show that $\bar{A} = A$ iff A is closed.