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N14. Bilinear forms

1. Definition of bilinear form.

Let V denote a vector space over $\mathbb{F}$.

Def 14.1 A function $B: V \times V \to \mathbb{F}$ is called a bilinear form if

1) $B(au+bv, w) = aB(u,w) + bB(v,w)$
2) $B(w, au+bv) = aB(w,u) + bB(w,v)$

Ex 14.1 a)

$$B(u,v) = \sum_{i,j=1}^{n} a_{ij} x_i y_j, \quad u,v \in \mathbb{F}^n,$$
$$u = (x_1, \dots, x_n)$$
$$v = (y_1, \dots, y_n)$$

b)
$$B(f,g) = \int_0^1 K(t) f(t) g(t)\,dt, \quad K, f, g \in C[0,1]$$

c) $B(u,v) = f(u)g(v)$, $u,v \in V$ and $f, g \in V^* = \mathcal{L}(V, \mathbb{F})$.

d) Let V be an inner product space over $\mathbb{R}$.
$B(u,v) = \langle u, v \rangle$, $u,v \in V$.

Let $e_1, \dots e_n$ be a basis in V. Let

$$v = x_1 e_1 + \dots + x_n e_n$$
$$u = y_1 e_1 + \dots + y_n e_n$$

We compute

$$B(v,u) = B\left(\sum_{i=1}^n x_i e_i, u\right) = \sum_{i=1}^n x_i B(e_i, u) =$$
$$= \sum_{i=1}^n x_i B\left(e_i, \sum_{j=1}^n y_j e_j\right) = \sum_i \sum_j x_i y_j B(e_i, e_j)$$
$$= \sum_{i,j=1}^n x_i y_j a_{ij}, \qquad (1)$$

where $a_{ij} := B(e_i, e_j)$

<u>Def 14.2</u> The matrix $A = (a_{ij})_{i,j=1}^n$, where $a_{ij} = B(e_i, e_j)$ is called the matrix of B in the basis $e_1, \dots, e_n$.
(A is also called the Gram matrix of B)

From (1):
$$B(v,u) = M_v^T A M_u, \qquad (2)$$
where $M_v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $M_u = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

2. Change of basis

Let $e = (e_1, \dots, e_n)$, $e' = (e_1', \dots, e_n')$ be bases in V.

Let $Q_{ee'}$ be a change-of-basis matrix from e to e', i.e.

$$e' = e\, Q_{ee'}$$

or

$$e'_j = \sum_{i=1}^{n} q_{ij} e_i \,, \quad \text{where } Q_{ee'} = (q_{ij})_{i,j=1}^{n}$$

we recall, that

$$M_v^e = Q_{ee'} M_v^{e'} \qquad (3)$$

Combining (2) and (3), we obtain

$$B(v,u) = (M_v^e)^T A\, M_u^e = \left(Q_{ee'} M_v^{e'}\right)^T A \left(Q_{ee'} M_u^{e'}\right) =$$

$$= \left(M_v^{e'}\right)^T Q_{ee'}^T A\, Q_{ee'} M_u^{e'}$$

<u>Th 14.1</u> Let A^e be a matrix of B in the basis e. Then the matrix of B in the basis e' is given by

$$A^{e'} = Q_{ee'}^T A^e Q_{ee'},$$

where $Q_{ee'}$ is the change-of-basis matrix from e to e'.

<u>Ex 14.2</u> Let $V = \mathbb{R}^2$, $B(v,u) = x_1y_1 + x_1y_2 + x_2y_2 + x_2y_2$,
$v = (x_1, x_2)$, $u = (y_1, y_2)$

Thus, $A^e = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is the matrix of B, that is,

$$B(v,u) = (x_1, x_2)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Let $e_1' = (1,-1) = 1e_1 - 1e_2$

$e_2' = (1,1) = 1\cdot e_1 + 1\cdot e_2$

in this case,

$$Q = Q_{ee'} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Thus, $$A^{e'} = Q^T A^e Q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}.$$

This means, that if

$$v = x_1' e_1' + x_2' e_2', \quad u = y_1' e_1' + y_2' e_2',$$

then

$$B(v,u) = 4 x_2' y_2'.$$

Def 14.3 The rank of the matrix of a bilinear form B is called the rank of B and is denoted by $\operatorname{rank} B$.

Exercise 14.1 Prove that the rank of B does not depent on the choise of a basis, i.e. $\operatorname{rank} A^e = \operatorname{rank} A^{e'}$.

Th 14.2 Let $\dim V = n$. Then the following conditions are equivalent

a) $\operatorname{rank} B = n$

b) $\forall\, v \neq 0 \quad \exists\, u \; : \; B(v,u) \neq 0$

c) $\forall\, u \neq 0 \quad \exists\, v \; : \; B(v,u) \neq 0$.

Def 14.4 If B satisfies one of conditions a) - c) of Th 14.2, B is called non-degenerate.

Th 14.3 Let B be a symmetric bilinear form $(A = A^T)$ in a real vector space V. Then there exist a basis of V in which the matrix of B is diagonal with only 1's, -1's, and 0's on the diagonal, namely, $\exists\, e_1, \dots, e_n$ - basis of V and $s \le r \le n$ such that

$$a_{ij} = B(e_i, e_j) = \begin{cases} 1, & i = j \le s \\ -1, & s < i = j \le r \\ 0, & i = j > r \\ 0, & i \neq j \end{cases}.$$

In this case $B(v,u) = x_1 y_1 + \dots + x_s y_s - x_{s+1} y_{s+1} - \dots - x_r y_r$

Th 14.4 (Sylvester's law of inertia) The number of 1's, -1's, and 0's in Th 14.3 is independent of the basis in which the matrix of B is written.

3. Quadratic forms

Let B be a symmetric bilinear form in V.

Def 14.5 The map from V to $\mathbb{F}$ defined as

$$v \mapsto B(v,v)$$

is called the quadratic form associated to a symmetric bilinear form B.

Remark Any quadratic form in a real vector space uniquelly determit the associated bilinear form by the equality

$$B(v,u) = \frac{1}{2}\left[B(v+u, v+u) - B(v,v) - B(u,u)\right]$$

Ex 14.2 a) $v = (x_1, \dots, x_n)$, $u = (y_1, \dots, y_n) \in \mathbb{R}^n$, $A = (a_{ij}) = A^T$.

$$B(v,u) = \sum_{ij} a_{ij} x_i y_j = (x_1, \dots, x_n) A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$B(v,v) = \sum_{ij} a_{ij} x_i x_j = (x_1 \dots x_n) A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

b) $v = (x_1, x_2)$, $u = (y_1, y_2)$

$$B(v,u) = a_{11} x_1 y_1 + a_{12} x_1 y_2 + a_{21} x_2 y_1 + a_{22} x_2 y_2$$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $A = A^T$.

$$B(v,v) = a_{11} x_1^2 + 2 a_{12} x_1 x_2 + a_{22} x_2^2$$

c) $B(v,v) = x_1^2 + 4x_1x_2 + x_2^2 - x_3^2 - x_2x_3$
in $\mathbb{R}^3$.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -1 \end{pmatrix}$$

<u>Th 14.5</u> For every quadratic form $B(v,v)$ in $\mathbb{R}^n$, there exists a basis in which

$$B(v,v) = x_1^2 + \ldots + x_s^2 - x_{s+1}^2 - \ldots - x_r^2.$$

Moreover, the numbers s and r are given uniquelly by the quadratic form.

<u>Ex 14.3</u> $B(v,v) = x_1^2 + 2x_1x_2 - 4x_1x_3 + 4x_3^2 -$
$- 6x_2x_3 =$

$$= \left(x_1^2 + 2\cdot x_1x_2 - 2\cdot x_1\cdot 2x_3 - 2\cdot x_2 2x_3 + x_2^2 + 4x_3^2\right) + 2x_2 2x_3 - x_2^2 - 6x_2x_3 =$$

$$= (x_1 + x_2 - 2x_3)^2 - 2x_2x_3 - x_2^2 =$$

$$= (x_1 + x_2 - 2x_3)^2 - \left(x_2^2 + 2x_2x_3 + x_3^2\right) + x_3^2 = (x_1 + x_2 - 2x_3)^2 - (x_2 + x_3)^2 + x_3^2$$

Taking
$$\begin{cases} z_1 = x_1 + x_2 - 2x_3 \\ z_2 = x_3 \\ z_3 = x_2 + x_3 \end{cases}$$
we have
$$B(v,v) = z_1^2 + z_2^2 - z_3^2$$

(8)

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{Q_{ee'}^{-1}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

in e' in e

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

Thus, $Q_{ee'} = \begin{pmatrix} 1 & 3 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Consequently

$$\left.\begin{aligned} e_1' &= 1e_1 + 0e_2 + 0e_3 = (1,0,0) \\ e_2' &= 3e_1 - 1e_2 + 1e_3 = (3,-1,1) \\ e_2' &= -1e_1 - 1e_2 + 0e_3 = (-1,1,0) \end{aligned}\right\} \text{new basis}$$

$B(v,v) = z_1^2 + z_2^2 - z_3^2$, where

$v = (z_1, z_2, z_3)$ in basis e'.

Def 14.6 • A quadratic form B is <u>positive definite</u> if $B(v,v) > 0 \quad \forall v \neq 0$.

• A quadratic form B is negative definite if $B(v,v) < 0 \quad \forall v \neq 0$.

Let $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ - & - & - \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$ be the matrix of B in some basis. The numbers

$$M_1 = a_{11}, \quad M_2 := \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \dots, \begin{vmatrix} a_{11} & \dots & a_{1n} \\ - & - & - \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ - & - & - & - & - \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

are called the principal minors of $B(v,v)$.

<u>Th 14.6</u> • The quadratic form B is positive definite if all principal minors are strictly positive, i.e. $M_1 > 0, M_2 > 0, \dots, M_n > 0$

• The quadratic form B is negative definite if $M_1 < 0, M_2 > 0, M_3 < 0, M_4 > 0, \dots$