(1)

Lecture N13. Unitary operators.

1. Definition and some properties.

Let V and W be inner product vector spaces over the same field $\mathbb{F}$.

Def 13.1 We say that $T \in \mathcal{L}(V,W)$ preserves inner products if

$$\langle Tu, Tv \rangle = \langle u, v \rangle \quad \forall u, v \in V$$

We also call T an isomorphism of the inner product space V onto W.

Note, that if T preserves inner product, then $\|Tv\| = \|v\|$.

Th 13.1 Let V and W be a finite-dimensional inner product spaces over $\mathbb{F}$ and $\dim V = \dim W$. If $T \in \mathcal{L}(V,W)$, then the following conditions are equivalent:

1) T preserves inner products;
2) T is an inner product space isomorphism, i.e. it is invertible and preserves inner products;
3) T carries some (and then every) orthonormal basis of V onto an orthonormal basis of W.

②

Proof See proof of Th 10 p. 300 [1]. ∎

Corollary 13.1 Let V and W be finite-dim. inner product spaces over $\mathbb{F}$. Then V and W are isomorphic iff they have the same dimension.

Proof. Let $e_1, \dots, e_n$ be an orthonormal basis in V and $e_1', \dots, e_n'$ be an orthonormal basis in W. We take $T \in \mathcal{L}(V, W)$ which is defined by the equality

$$Te_i = e_i'.$$

Then T is isomorphism of V onto W.

Ex 13.1 a) Let V be an n-dim inner product vector space over $\mathbb{F}$ and let $e_1, \dots, e_n$ be an orthonormal basis in V. We define the linear map

$$Tv = T(a_1e_1 + \dots + a_ne_n) = (a_1, \dots, a_n),$$

where $a_i = \langle v, e_i \rangle$, determines an isomorphism of V onto $\mathbb{F}^n$.

b) Let $V = \mathbb{F}^n$, $W = \mathbb{F}^{n \times 1}$ with an inner product $\langle X, Y \rangle = Y^* X =$

$$= (\bar{y}_1 \cdots \bar{y}_n) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Then the map $T(x_1, \dots, x_n) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is an isomorphism of $\mathbb{F}^n$ onto $\mathbb{F}^{n\times 1}$.

Th 13.2 $T \in \mathcal{L}(V, W)$ preserves inner products iff $\|Tv\| = \|v\|$ for all $v \in V$.

Proof The theorem follows from equalities

$$\langle u, v\rangle = \frac{1}{4}\|v+u\|^2 - \frac{1}{4}\|v-u\|^2 \quad \text{if } \mathbb{F}=\mathbb{R}$$

$$\langle u, v\rangle = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2 + \frac{i}{4}\|u+iv\|^2 - \frac{i}{4}\|u-iv\|^2 \quad \text{if } \mathbb{F}=\mathbb{C}.$$

∎

Def 13.2 $T \in \mathcal{L}(V)$ is called a unitary operator if T preserves inner products (and is map from V to V).

Th 13.3 Let $T \in \mathcal{L}(V)$. Then T is unitary iff $T^*T = TT^* = I$, i.e. T is invertible and $T^{-1} = T^*$.

Proof We assume that T is unitary. Then $\langle Tv, u\rangle = \langle Tv, TT^{-1}u\rangle = \langle v, T^{-1}u\rangle$ for all $v, u \in V$. (Here T is invertible by Th. 13.1). Thus, $T^* = T^{-1}$.

(4) Now, let $T^* = T^{-1}$! Then

$$\langle Tv, Tu \rangle = \langle v, T^* T u \rangle =$$

$$= \langle v, T^{-1} T u \rangle = \langle v, u \rangle.$$

$\blacksquare$

Def 13.3 • A ~~complex~~ matrix $A \in \mathbb{F}^{n \times n}$ is called unitary, if $A^* A = I$.

• A real matrix $A \in \mathbb{R}^{n \times n}$ is called orthogonal if $A^T A = I$.

Th 13.4 $T \in \mathcal{L}(V)$ is unitary iff its matrix M_T in some (and then every) orthonormal basis is unitary.

Ex 13.2 a) The matrix $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where $a^2 + b^2 = 1$, is orthogonal. Indeed,

$$A^T A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

b) $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is the matrix in the standard basis of a linear map T_θ that is a rotation through the angle $\theta \in \mathbb{R}$.

A_θ is also orthogonal.

Th 13.5 (i) Let $A \in \mathbb{F}^{n\times n}$ be self-adjoint. Then there exists a unique matrix P such that $P^{-1}AP$ is diagonal.

(ii) If A is a real symmetric matrix, then there exists a real orthogonal matrix P such that $P^{-1}AP$ is diagonal.

Th 13.6 If A is unitary, then its rows (columns) form an orthonormal basis in $\mathbb{F}^n$

Proof Let $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ - & - & - \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$ and

$$A^{-1} = (\bar{A})^T = \begin{pmatrix} \bar{a}_{11} & \dots & \bar{a}_{n1} \\ - & - & - \\ \bar{a}_{1n} & & \bar{a}_{nn} \end{pmatrix}.$$

Take $e_i = (a_{i1}, \dots, a_{in})$.

Then

$$AA^{-1} = \begin{pmatrix} \langle e_1, e_1\rangle & \langle e_1, e_2\rangle \dots & \langle e_1, e_n\rangle \\ - & - & - \\ \langle e_n, e_1\rangle & \langle e_n, e_2\rangle \dots & \langle e_n, e_n\rangle \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ - & - & - & - \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

This implies that $\langle e_i, e_j\rangle = \delta_{ij}$.

2. Normal operators

<u>Def. 13.4</u> A linear map $T \in \mathcal{L}(V)$ is called <u>normal</u> if $TT^* = T^*T$.

<u>Th 13.7</u> (Spectral theorem). Let V be a finite-dim. inner product space over $\mathbb{C}$ and $T \in \mathcal{L}(V)$. Then T is normal iff there exists an orthonormal basis in V consisting of eigenvectors of T.

<u>Proof</u> See proof of Th 11.3.1 p. 120 [2].

<u>Corollary 13.1</u> Let $T \in \mathcal{L}(V)$ be a normal operator and V be finite-dim inner product space over $\mathbb{C}$. Let also $\lambda_1, \dots, \lambda_m$ be distinct eigenvalues of T. Then

1) $V = V_{\lambda_1} \oplus \dots \oplus V_{\lambda_m}$, where

$$V_{\lambda_i} = \{v : Tv = \lambda_i v\} = \ker(T - \lambda_i I)$$

2) if $i \neq j$, then $V_{\lambda_i} \perp V_{\lambda_j}$, i.e. for all $v \in V_{\lambda_i}$ and $u \in V_{\lambda_j}$

$$\langle v, u \rangle = 0.$$

⑦

<u>Remark 13.1</u> Th 13.7. tell us that $T \in L(V)$ is normal iff M_T is diagonal with respect to an orthonormal basis $e_1, \dots, e_n$ in V, i.e. there exists a unitary matrix U such that

$$U M_T \underbrace{U^*}_{= U^{-1}} = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

<u>Remark 13.2</u> The diagonal decomposition allow us to easily compute powers and the functions of matrices. Namely, let

$$A = U D U^{-1}, \text{ where } D = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

Then

$$A^n = (UDU^{-1})^n = U D^n U^{-1} = U \begin{pmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_n^n \end{pmatrix} U^{-1}.$$

Indeed, $A^2 = U D \cancel{U^{-1}} \cancel{U} D U^{-1} = U D^2 U^{-1}$.

Thus, we can define

$$f(A) = U \begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix} U^{-1}. \qquad (12.1)$$

Let us explain (12.1).

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Assume that we want to define e^A.

Then $e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = U\left(\sum_{k=0}^{\infty} \frac{1}{k!} D^k\right)U^{-1} =$

$$= U \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_n^k \end{pmatrix} U^{-1} = U\begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} U^{-1}.$$

Ex. 13.2 $\quad A = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}$.

- A is self-adjoint and thus normal.

Solving the characteristic equation

$$|A - \lambda I| = 0,$$

we obtain $\lambda_1 = 1$, $\lambda_2 = 4$.

The corresponding eigen values are

$$v_1 = (-1-i, 1), \quad v_2 = (1+i, 2)$$

- Take $\quad e_1 = \left(\frac{-1-i}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $\quad e_2 = \left(\frac{1+i}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$.

Then,

$$A = \underbrace{\begin{pmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{pmatrix}}_{=U} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \underbrace{\begin{pmatrix} \frac{-1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1-i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix}}_{U^{-1}}$$

Then

$$e^A = U\begin{pmatrix} e & 0 \\ 0 & e^4 \end{pmatrix}U^{-1} = \frac{1}{3}\begin{pmatrix} 2e+e^4 & e^4-e+i(e^4-e) \\ e^4-e+i(e-e^4) & e+2e^4 \end{pmatrix}$$

[1] K. Hoffman, R. Kunze "Linear Algebra"

[2] I. Lankham,.... "Linear Algebra as an Introduction to abstract Math."