Lecture 12. Adjoint operators

1. Dual space.

Let V be an finite dimensional inner product space over $\mathbb{F}$. We fix $u \in V$ and consider the map from V to $\mathbb{F}$:

$$f_u(v) = \langle v, u \rangle. \qquad (12.1)$$

By properties of inner product, f_u is a linear map or a linear functional on V (a map from V to scalars $\mathbb{F}$ is also called a functional on V)

Th 12.1 Let V be a finite-dimensional inner product space, and f be a linear functional on V. Then there exists a unique vector $u \in V$ such that

$$f(v) = \langle v, u \rangle$$

for all $v \in V$.

Proof. Let $e_1, \dots, e_n$ be an orthonormal basis in V. Then

$$f(v) = f(\langle v, e_1 \rangle e_1 + \dots + \langle v, e_n \rangle e_n) =$$

$$= \langle v, e_1 \rangle \overline{\overline{f(e_1)}} + \dots + \langle v, e_n \rangle \overline{\overline{f(e_n)}} =$$

$$= \langle v, \underbrace{\bar{f}(e_1)e_1 + \dots + \bar{f}(e_n)e_n}_{=:u} \rangle .$$

If $f(v) = \langle v, u_1 \rangle = \langle v, u_2 \rangle \quad \forall v \in V$.

Then $0 = f(v) - f(v) = \langle v, u_1 \rangle - \langle v, u_2 \rangle = \langle v, u_1 - u_2 \rangle$

Taking $v = u_1 - u_2$:

$$\langle u_1 - u_2, u_1 - u_2 \rangle = \|u_1 - u_2\|^2 = 0 .$$

Thus, $u_1 = u_2$. This proves the uniqueness. ∎

<u>Remark 12.1</u> Th. 12.1 is called Riesz representation theorem.

<u>Def 12.1</u> The set of all linear functionals on V is called the dual space of V and is denoted by V^*. So $V^* = \mathcal{L}(V, \mathbb{F})$.

<u>Remark 12.2</u> By the Riesz representation theorem, V^* contains only functionals of the form (12.1).

2. Adjoint operators

<u>Th 12.2</u> Let $T \in \mathcal{L}(V)$. Then there exists a unique $T^* \in \mathcal{L}(V)$ such that

$$\langle Tv, u \rangle = \langle v, T^* u \rangle ,$$

for all $u, v \in V$.

<u>Def 12.2.</u> The operator T^* is called <u>adjoint of T</u>.

③ Proof.

Let $u \in V$. Then the map

$$v \to \langle Tv, u \rangle$$

is a linear functional on V. By Th 12.1, there exists a unique $u' \in V$ such that

$$\langle Tv, u \rangle = \langle v, u' \rangle.$$

Let T^* denotes the map $u \to u'$, i.e.

$$T^* u = u'.$$

We have to prove that T^* is linear.

1) Let $u_1, u_2 \in V$. For each $v \in V$

$$\langle v, T^*(u_1+u_2) \rangle = \langle Tv, u_1 + u_2 \rangle =$$

$$= \langle Tv, u_1 \rangle + \langle Tv, u_2 \rangle = \langle v, T^*u_1 + T^*u_2 \rangle.$$

Thus, $T^*(u_1+u_2) = T^*u_1 + T^*u_2$.

2) similarly, $T^*(au) = aT^*u$. ∎

Th 12.3 Let $e_1, \dots, e_n$ be an orthonormal basis in V. Then the entries of the map M_T of a linear operator $T \in \mathcal{L}(V)$ in basis $e_1, \dots, e_n$ are defined by

$$(12.2) \qquad a_{ij} = \langle Te_j, e_i \rangle, \quad i,j = 1, \dots, n.$$

Proof. We recall that the j^{th} column of M_T contains the coordinates of the vector Te_j in the basis $e_1, \dots, e_n$, that is,

$$Te_j = \underbrace{\langle Te_j, e_1 \rangle}_{a_{1j}} e_1 + \dots + \underbrace{\langle Te_j, e_n \rangle}_{a_{nj}} e_n .$$

$\blacksquare$

Th 12.4 Let $T \in \mathcal{L}(V)$. Then in any orthonormal basis of V, the matrix M_{T^*} of T^* is the complex conjugate transpose of the matrix M_T of T.

Notation: The complex conjugate transpose of matrix A is denoted A^*.

Proof. The proof immediately follows from (12.2). Indeed, the entries of M_{T^*} is given by the formula

$$b_{ij} = \langle T^* e_j, e_i \rangle = \overline{\langle e_i, T^* e_j \rangle} =$$
$$= \overline{\langle Te_i, e_j \rangle} = \overline{a_{ji}},$$

where a_{ij} are entries of M_T.

Th 12.5 Let $T, S \in \mathcal{L}(V)$, $a \in \mathbb{F}$. Then

1) $(T+S)^* = T^* + S^*$
2) $(aT)^* = \bar{a}T^*$
3) $(TS)^* = S^* T^*$
4) $(T^*)^* = T$

3. Self-adjoint operators.

Def 12.3 • If $T \in \mathcal{L}(V)$ satisfies $T = T^*$, then T is called self-adjoint (or Hermitian)

• A matrix A is called self-adjoint if $A = A^*$.

Remark 12.3 T is self-adjoint if its matrix M_T (in any orthonormal basis) is self-adjoint.

Th 12.6 Let T, S be self-adjoint linear maps on V. Then the following operators are also self-adjoint:

$$T^2,\ T+S,\ aT,\ TS+ST$$

(TS is not nesessarily adjoint).

Th 12.7 Let $T \in \mathcal{L}(V)$ be a self-adjoint. Then each eigenvalue of T is real, and eigenvectors which correspond distinct eigenvalues are orthogonal.

Proof. Let λ be an eigenvalue of T. Then for some non-zero vector v

$$Tv = \lambda v.$$

Then

$$\lambda \|v\|^2 = \lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle Tv, v \rangle = \langle v, Tv \rangle =$$

$$= \langle v, \lambda v \rangle = \bar{\lambda} \langle v, v \rangle = \bar{\lambda} \|v\|^2 \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}.$$

Let next $Tv_1 = \lambda_1 v_1$, $Tv_2 = \lambda_2 v_2$ and $\lambda_1 \neq \lambda_2$. We compute

$$\lambda_1 \langle v_1, v_2 \rangle = \langle T v_1, v_2 \rangle = \langle v_1, Tv_2 \rangle =$$

$$= \bar{\lambda}_2 \langle v_1, v_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle$$

Since $\lambda_1 \neq \lambda_2$, $\langle v_1, v_2 \rangle = 0$.

Th 12.8 Let $T \in \mathcal{L}(V)$ be self-adjoint.

(i) Then there is an orthogonal basis of V, each vector of which is an eigenvector of T corresponding a real eigenvalue of T.

(ii) There exists an orthonormal basis in which M_T as a diagonal form, where each entry is real.

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<u>Ex. 12. 9</u> $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = (1-\lambda-2)(1-\lambda+2)$$
$$= (-1-\lambda)(3-\lambda) = 0$$

$\lambda_1 = -1$
$\lambda_2 = 3$

$\lambda_1 = -1$: $\begin{cases} (1+1)x + 2y = 0 \\ 2x + 2y = 0 \end{cases} \Rightarrow x = -y$

Hence $v_1 = (1, -1)$, $e_1 = \dfrac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$\lambda_2 = 3$: $\begin{cases} -2x + 2y = 0 \\ 2x - 2y = 0 \end{cases} \Rightarrow x = y$

$v_2 = (1, 1)$; $e_2 = \dfrac{v_2}{\|v_2\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

e_1, e_2 is an orthonormal basis in $\mathbb{R}^2$.

$\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$ - matrix of the linear operator $v \to Av$ in basis e_1, e_2.