Lecture 11. Orthogonal projections

1. Orthogonal projection.

Let V be a vector space over $\mathbb{F}$.
We recall that $u, v \in V$ are orthogonal if $\langle u, v\rangle = 0$.

- Let $U \subseteq V$ be a subset (not necessarily a subspace) of V.

Def 11.1 The set

$$U^{\perp} = \{ v \in V : \langle u, v \rangle = 0 \quad \forall u \in U \}$$

is called the orthogonal complement of U.

- Exercise 11.1 a) Show that $U^{\perp}$ is always a subspace of V.
  b) Show that $\{0\}^{\perp} = V$ and $V^{\perp} = \{0\}$.
  c) Show that $U_1 \subset U_2$ implies $U_2^{\perp} \subset U_1^{\perp}$.

Ex 11.1 Let $U = \{(1,0,0), (0,1,0)\} \subseteq \mathbb{R}^3$

Then $U^{\perp} = \{ \underset{(x,y,z)}{\overset{\shortparallel}{v}} \in \mathbb{R}^3 : \langle v, (1,0,0)\rangle = 1\cdot x + 0\cdot y + 0 \cdot z = 0,$
$\langle v, (0,1,0)\rangle = y = 0 \} =$

$= \{ v = (x,y,z) : x = 0, y = 0\} = \{ v = (0,0,z) : z \in \mathbb{R} \}.$

Lemma 11.1 Let U be a subset of V. Then $(\operatorname{span} U)^\perp = U^\perp$.

Proof Since $U \subseteq \operatorname{span} U$, $(\operatorname{span} U)^\perp \subseteq U^\perp$, by Ex 11.1. Next, we show that

$$U^\perp \subseteq (\operatorname{span} U)^\perp.$$

We take $v \in U^\perp$. Then $\forall u \in U$ $\langle v, u\rangle = 0$. If $u \in \operatorname{span} U$, then

$$u = a_1 u_1 + \dots + a_n u_n$$

where $a_i \in \mathbb{F}$ and $u_i \in U$. Thus,

$$\langle u, v\rangle = a_1 \underbrace{\langle u_1, v\rangle}_{=0} + \dots + a_n \underbrace{\langle u_n, v\rangle}_{=0} = 0.$$

This implies that $v \in (\operatorname{span} U)^\perp$. ∎

We recall that V is a direct sum of U_1 and U_2 if for all $v \in V$ there exist unique $u_1 \in U_1$ and $u_2 \in U_2$ such that

$$v = u_1 + u_2 \quad (\text{Notation } V = U_1 \oplus U_2)$$

(see Def 23.3, Math 1)

Proposition 23.2 (Math 1) Let U_1 and U_2 be vector subspaces of V. Then $V = U_1 \oplus U_2$ iff

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1) $V = U_1 + U_2$, that is,
for each $v \in V$ there exist $u_1 \in U_1$ and $u_2 \in U_2$ such that
$$v = u_1 + u_2$$
2) $U_1 \cap U_2 = \{0\}$.

Th 11.1 If U is a subspace of V, then
$$V = U \oplus U^\perp$$

Proof. We check the conditions of Prp 23.2.

1) Let $e_1, \dots, e_m$ be an orthonormal basis in U. For v we consider
$$v = \overbrace{\langle v, e_1 \rangle e_1 + \dots + \langle v, e_m \rangle e_m}^{u_1} +$$
$$+ \underbrace{v - \langle v, e_1 \rangle e_1 - \langle v, e_m \rangle e_m}_{u_2}$$
It is clear that $u_1 \in U$. We show that $u_2 \in U^\perp$. Indeed, for $e_j \in U$
$$\langle u_2, e_j \rangle = \langle v, e_j \rangle - \langle v, e_j \rangle = 0.$$
So, $u_2 \in \{e_1, \dots, e_m\}^\perp \overset{\text{Lemma 11.1}}{=} (\operatorname{span}\{e_1, \dots, e_m\})^\perp =$
$$= U^\perp.$$

2) Let $u \in U \cap U^\perp$, then
$$\|u\|^2 = \langle \underset{U}{u}, \underset{U^\perp}{u} \rangle = 0 \quad \Rightarrow u = 0.$$

Ex 11.2 a) Let $V = \{(x,y,0) : x,y \in \mathbb{R}\} \subseteq \mathbb{R}^3$

$V^\perp = \{(0,0,z) : z \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

Then,

$$\mathbb{R}^3 = V \oplus V^\perp.$$

b) $V = \operatorname{span}\{v_1 = (1,1,0,0),\ v_2 = (1,1,1,1)\}$

$= \{a v_1 + b v_2 = (a+b, a+b, b, b),\ a,b \in \mathbb{R}\}$

$V^\perp = \{v \in \mathbb{R}^4 : \langle v, v_1\rangle = 0,\ \langle v, v_2\rangle = 0\}$

Let $v = (x_1, x_2, x_3, x_4)$. Then

$$(11.1) \quad \begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 + x_3 + x_4 = 0 \end{cases} \qquad \left(\begin{array}{cc|cccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right)$$

Hence $(-1,1,0,0)$, $(0,0,-1,1)$ – fundamental system of solutions of (11.1).

So,

$$V^\perp = \operatorname{span}\{(-1,1,0,0), (0,0,-1,1)\} =$$

$$= \{(-a, a, -b, b),\ a,b \in \mathbb{R}\}.$$

Moreover,

$$\mathbb{R}^4 = V \oplus V^\perp.$$

Th. 11.2 Let U be a subset of V. Then
$$\operatorname{span} U = (U^\perp)^\perp.$$
in particular, if U is a subspace of V, then
$$U = (U^\perp)^\perp.$$

Proof. Let U be a vector subspace of V.
We first show that $U \subset (U^\perp)^\perp$. Take $u \in U$.
Then for all $v \in U^\perp$ $\langle u, v \rangle = 0$. Thus,
$$u \in (U^\perp)^\perp.$$
Next we show that
$$(U^\perp)^\perp \subseteq U.$$
We assume, that $0 \neq v \in (U^\perp)^\perp$ and $v \notin U$.
By th. 11.1.
$$v = \underset{\substack{\cap \\ U}}{u_1} + \underset{\substack{\cap \\ U^\perp}}{u_2}.$$
Since $v \notin U$, we have $u_2 \neq 0$.
Moreover, $\langle u_2, v \rangle = \langle u_2, u_2 \rangle \neq 0$
but then v is not in $(U^\perp)^\perp$, which contradict our initial assumption.
Hence $(U^\perp)^\perp \subseteq U$.

$\blacksquare$

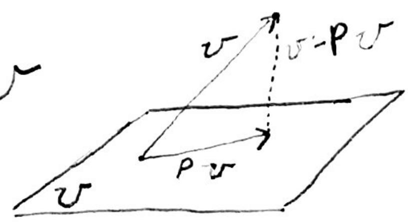
By Th 11.1, we have the decomposition
$$V = \mathcal{U} \oplus \mathcal{U}^{\perp}$$
for every subspace $\mathcal{U}$ of V.

<u>Def 11.2</u> The map $P_{\mathcal{U}} : V \to V$, defined as $P_{\mathcal{U}} v = u_1$, where u_1 is taken from the decomposition
$$v = u_1 + u_2, \quad u_1 \in \mathcal{U}, \quad u_2 \in \mathcal{U}^{\perp}$$
is called an orthogonal projection.

Let $e_1, \dots, e_m$ be an orthonormal basis in $\mathcal{U}$. Then according to the proof of Th 1

$$(11.2) \qquad P_{\mathcal{U}} v = \langle v, e_1 \rangle e_1 + \dots + \langle v, e_m \rangle e_m .$$

<u>Th 11.3</u> Let $\mathcal{U} \subset V$ be a subspace of V and $v \in V$. Then
$$\| v - P_{\mathcal{U}} v \| \le \| v - u \| \quad \forall u \in \mathcal{U}$$
Moreover, the equality holds iff $u = P_{\mathcal{U}} v$.



<u>Proof</u> Let $u \in \mathcal{U}$ and $P := P_{\mathcal{U}}$. Then
$$\| v - Pv \|^2 \le \| \underbrace{v - Pv}_{\in \mathcal{U}^{\perp}} \|^2 + \| \underbrace{\underbrace{Pv}_{\in \mathcal{U}} - \underbrace{u}_{\in \mathcal{U}}}_{\in \mathcal{U}} \|^2 \overset{\text{Pythagorem Th}}{=}$$
$$= \| v - Pv + Pv - u \|^2 = \| v - u \|^2 .$$
The equality holds iff $\| Pv - u \|^2 = 0 \Rightarrow Pv = u$. ∎