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Lecture N10. Orthonormal basis

1. Inner product and norm

We recall the definitions:

Def 10.1 A map $\langle\cdot,\cdot\rangle : V\times V \to \mathbb{F}$ satisfying

1) $\langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle$ and
   $\langle au, v\rangle = a\langle u,v\rangle \quad \forall u,v,w\in V,\ a\in\mathbb{F}$
2) $\langle v,v\rangle \geqslant 0 \quad \forall v\in V$
3) $\langle v,v\rangle = 0$ iff $v=0$
4) $\langle u,v\rangle = \overline{\langle v,u\rangle} \quad \forall u,v\in V$

is called an inner product on V.

Def 10.2 A map $\|\cdot\| : V\to [0,+\infty)$ satisfying

1) $\|av\| = |a|\,\|v\| \quad \forall v\in V,\ a\in\mathbb{F}$
2) $\|v\| = 0 \Leftrightarrow v=0$
3) $\|u+v\| \leq \|u\| + \|v\|, \quad v,w\in V$

is called a norm on V

Ex 10.1 Let $V=\mathbb{C}^n$, $\langle u,v\rangle = a_1\bar{b}_1 + \ldots + a_n\bar{b}_n$.

Then $\langle\cdot,\cdot\rangle$ is an inner product on $\mathbb{C}^n$.

Indeed, $u=(a_1,\ldots,a_n)$
$v=(b_1,\ldots,b_n)$
$w=(d_1,\ldots,d_n)$

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1) $\langle u+v, w\rangle = (a_1+b_1)\bar{c}_1 + \dots + (a_n+b_n)\bar{c}_n =$

$= a_1\bar{c}_1 + \dots + a_n\bar{c}_n + b_1\bar{c}_1 + \dots + b_n\bar{c}_n =$

$= \langle u, w\rangle + \langle v, w\rangle.$

$\langle au, v\rangle = a\langle u, v\rangle$ - similarly

2) $\langle v, v\rangle = b_1\bar{b}_1 + \dots + b_n\bar{b}_n = |b_1|^2 + \dots + |b_n|^2 \geq 0$

3) $\langle v, v\rangle = 0 \iff v = (0, \dots, 0)$

4) $\langle u, v\rangle = \overline{\overline{a_1\bar{b}_1 + \dots + a_n\bar{b}_n}} = \overline{\bar{a}_1 b_1 + \dots + \bar{a}_n b_n} =$

$= \overline{\langle v, u\rangle}.$

<u>Def 10.3</u> A vector space V over $\mathbb{F}$ with an inner product $\langle \cdot, \cdot \rangle$ is called an <u>inner product space</u>.

Let V be an inner product space.
Define

$$\|v\| = \sqrt{\langle v, v\rangle}. \qquad (10.1)$$

Last time we show

$$|\langle v, u\rangle| \leq \|v\| \cdot \|u\|$$

– <u>Cauchy - Schwarz inequality</u>.

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Th 10.1 (Triangle inequality). Let $\|\cdot\|$ be defined by (10.1). Then for all $u, v \in V$

$$\|u+v\| \leq \|u\| + \|v\|.$$

Proof HW.

Remark 10.1. For all $u, v, w \in V$, $a \in \mathbb{F}$

$$\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$$
$$\langle u, av\rangle = \bar{a}\langle u, v\rangle.$$

Indeed, $\langle u, v+w\rangle = \overline{\langle v+w, u\rangle} = \overline{\langle v, u\rangle + \langle w, u\rangle} =$

$= \langle u, v\rangle + \langle u, w\rangle.$

The same for the second equality.

Ex 10.2 $V = \mathbb{R}^n$, $\langle u, v\rangle = a_1 b_1 + \dots + a_n b_n$

Then $\|u\| = \sqrt{\langle u, u\rangle} = \sqrt{a_1^2 + \dots + a_n^2}$.

Cauchy - Schwarz inequality:

$$|a_1 b_1 + \dots + a_n b_n| \leq \sqrt{a_1^2 + \dots + a_n^2} \cdot \sqrt{b_1^2 + \dots + b_n^2}.$$

Triangle inequality:

$$\sqrt{(a_1+b_1)^2 + \dots + (a_n+b_n)^2} \leq \sqrt{a_1^2 + \dots + a_n^2} + \sqrt{b_1^2 + \dots + b_n^2}.$$

Corollary 10.1 The function $\|\cdot\|$ defined by (10.1) is a norm on the inner product space V.

2. Orthonormal bases

Def 10.4 Two vectors $u, v \in V$ are orthogonal ($u \perp v$) if $\langle u, v \rangle = 0$.

Th 10.2 (Pythagorem Theorem). If $u, v \in V$ and $u \perp v$, then

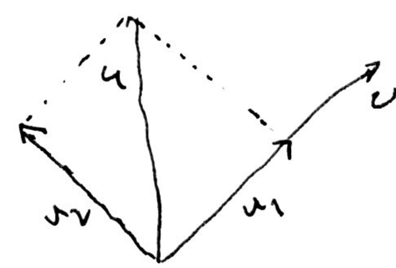
$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

Proof. H.W

Let u, v be two vectors from V and $v \neq 0$. Then u can be uniquelly written as

$$u = u_1 + u_2, \text{ where}$$

$u_1 = av$, $u_2 \perp v$



Indeed, let $u_1 = av$. Then

$u_2 = u - av$

$$0 = \langle u - av, v \rangle = \langle u, v \rangle - a \langle v, v \rangle$$

Hence $a = \dfrac{\langle u, v \rangle}{\|v\|^2}$.

So,

$$u = \frac{\langle u, v \rangle}{\|v\|^2} v + \left(u - \frac{\langle u, v \rangle}{\|v\|^2} v \right).$$

Def 10.5 • A list of non-zero vectors $e_1, \dots, e_n$ in V is called orthogonal if

$$\langle e_i, e_j \rangle = 0 \quad \forall i \neq j$$

• $e_1, \dots, e_n$ is called orthonormal if

$$\langle e_i, e_j \rangle = \delta_{ij}.$$

Th 10.3 Every orthogonal list of non-zero vectors in V is linearly independent

Proof Let $e_1, \dots, e_m$ be orthogonal. Consider

$$a_1 e_1 + \dots + a_m e_m = 0$$

For any i we consider

$$\langle a_1 e_1 + \dots + a_i e_i + \dots + a_m e_m, e_i \rangle = \langle 0, e_i \rangle = 0$$

$$\|$$

$$a_1 \underbrace{\langle e_1, e_i \rangle}_{=0} + \dots + a_i \underbrace{\langle e_i, e_i \rangle}_{=\|e_i\|^2 > 0} + \dots + a_m \underbrace{\langle e_m, e_i \rangle}_{=0} = 0$$

$\Rightarrow a_i = 0$ for all $i = 1, \dots, m$.

Def 10.6 An orthonormal basis of V is a list of orthonormal vectors that is basis in V.

Th 10.4 Let $e_1, \dots, e_n$ be an orthonormal basis for V. Then for all $v \in V$

$$v = \overbrace{\langle v, e_1 \rangle}^{a_1} e_1 + \dots + \overbrace{\langle v, e_n \rangle}^{=a_n} e_n = a_1 e_1 + \dots + a_n e_n$$

and $\|v\|^2 = \sum_{i=1}^{n} |\langle v, e_i \rangle|^2$

$\langle v, u \rangle = a_1 \bar{b}_1 + \dots + a_n \bar{b}_n$ if $u = b_1 e_1 + \dots + b_n e_n$.

⑥ The Gram-Schmidt orthogonalization procedure.

Th 10.5 If $v_1, \dots, v_m$ is a list of linearly independent vectors in an inner product space V, then there exists an orthonormal list of $(e_1, \dots, e_m)$ such that

$$\operatorname{span}\{v_1, \dots, v_k\} = \operatorname{span}\{e_1, \dots, e_k\}, \quad \forall k = 1 \dots n.$$

Proof We set $e_1 = \dfrac{v_1}{\|v_1\|}$.

Next

$$e_2 = \frac{v_2 - \langle v_2, e_1\rangle e_1}{\| v_2 - \langle v_2, e_1\rangle e_1 \|}$$

Then $\operatorname{span}\{e_1, e_2\} = \operatorname{span}\{v_1, v_2\}$.

Let $e_1, \dots, e_{k-1}$ be constructed, then

$$e_k = \frac{v_k - \langle v_k, e_1\rangle e_1 - \dots - \langle v_k, e_{k-1}\rangle e_{k-1}}{\| v_k - \langle v_k, e_1\rangle e_1 - \dots - \langle v_k, e_{k-1}\rangle e_{k-1} \|}.$$

Then $\langle e_k, e_i\rangle = 0$, $i = 1, \dots, k-1$ and

$$\operatorname{span}\{e_1, \dots, e_k\} = \operatorname{span}\{v_1, \dots, v_k\}.$$

Ex. 10.3 Take $v_1 = (1,1,0)$, $v_2 = (2,1,1)$ in $\mathbb{R}^3$.

Then

$$e_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,0)}{\sqrt{1^2+1^2+0^2}} = \frac{1}{\sqrt2}(1,1,0) = \left(\frac{1}{\sqrt2}, \frac{1}{\sqrt2}, 0\right)$$

Compute

$$\langle v_2, e_1\rangle = \frac{1}{\sqrt2}(2,1,1)\cdot(1,1,0) = \frac{3}{\sqrt2}$$

$$u_2 = v_2 - \langle v_2, e_1\rangle = (2,1,1) - \frac{3}{2}(1,1,0) = \frac{1}{2}(1,-1,2)$$

$$e_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{6}}(1,-1,2), \text{ since } \|u_2\| = \sqrt{\tfrac{1}{4}(1+1+4)} = \frac{\sqrt{6}}{2}.$$

Thus

$e_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $e_2 = \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$ - orthonormal system.

<u>Corollary 10.2</u> Every finite-dimensional inner product space has an orthonormal basis.

<u>Corollary 10.3</u> Every orthonormal list of vectors in V can be extended to an orthonormal basis of V.