Lecture N9. Inner product

1. Diagonalization of linear maps.

We recall that λ is called an eigenvalue of a linear operator $T: V \to V$ if

$$Tv = \lambda v \qquad (9.1)$$

for some non-zero vector $v \in V$.

The vector v in (9.1) is called an eigenvector corresponding to λ.

<u>Th 8.4</u> A linear map $T: V \to V$ is diagonalizable iff $\exists$ $v_1, \dots, v_n$ - basis in V such that $\forall i$ v_i are eigenvectors corresponding to λ_i.

In this case

$$M_T = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

in this basis

<u>Th 9.1</u> Let $\dim V = n$ and T have n distinct eigenvalues, then T is diagonalizable.

(2)

<u>Proof.</u> Let $\lambda_1, \dots, \lambda_n$ be distinct eigenvalues of T. Then there exists $v_1, \dots, v_n$ – eigenvectors corresponding to $\lambda_1, \dots, \lambda_n$. By Th 8.3, they are linearly independent. Thus, $v_1, \dots, v_n$ form a basis in V.

By Th 8.4. T is diagonalizable and

$$M_T = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ - & - & - & - \\ 0 & 0 & & \lambda_n \end{pmatrix}$$

in the basis $v_1, \dots, v_n$.

<u>Ex. 9.1</u> $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \in \mathbb{R}^{2\times 2}$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 =$$

$$= (1-\lambda-2)(1-\lambda+2) = (-1-\lambda)(3-\lambda) = 0$$

$\lambda_1 = -1, \ \lambda_2 = 3$

Next we find eigenvectors:

λ_1: $Av = \lambda_1 v \iff (A - \lambda_1 I)v$

$$\begin{pmatrix} 1+1 & 1 \\ 4 & 1+1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{cases} 2x + y = 0 \\ 4x + 2y = 0 \end{cases} \Rightarrow \begin{matrix} x = 1 \\ y = -2 \end{matrix} \quad v_1 = (1, -2)$$

λ_2:

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{cases} -2x + y = 0 \\ 4x - 2y = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 2 \end{cases}, \quad v_2 = (1, 2)$$

Thus, A is diagonal in basis $v_1 = (1,-2)$
$v_2 = (1,2)$.

Th 9.2 Let $\dim V = n$, $T: V \to V$ be a linear map, $\lambda_1, \dots, \lambda_m$ be distinct eigenvalues of T and

$$V_{\lambda_i} = \operatorname{Ker}(T - \lambda_i I) = \{v : Tv = \lambda_i v\}$$

be corresponding eigenspaces.
The following conditions are equivalent:

(i) T is diagonalizable

(ii) The characteristic polynomial of T is

$$\det(M_T - \lambda I) = (\lambda_1 - \lambda)^{n_1} \dots (\lambda_m - \lambda)^{n_m}$$

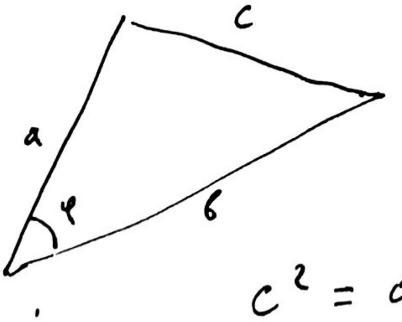
and $\dim V_{\lambda_i} = n_i$, $i = 1, \dots, n$

(iii) $\dim V_{\lambda_1} + \dots + \dim V_{\lambda_m} = n$.

2. Scalar product in $\mathbb{R}^3$ (in $\mathbb{R}^2$)

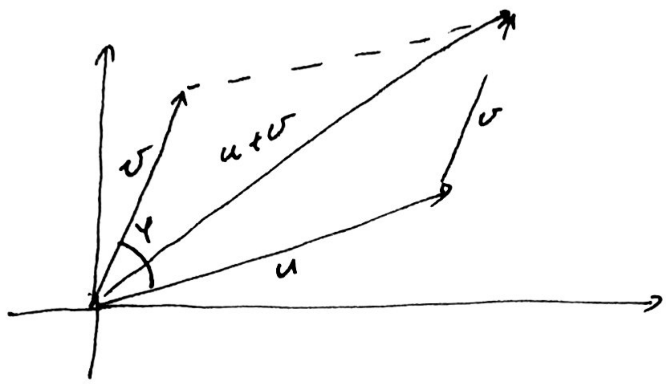
In this section we will work with the space $\mathbb{R}^2$. All results can be easily transfered to the case of $\mathbb{R}^3$.

Let us consider a triangle:



Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\varphi$

Now we consider two vectors



Then $|u+v|^2 = |v|^2 + |u|^2 + \overbrace{2|u||v|\cdot\cos\varphi}^{\text{scalar product}}$,
where φ is the angle between v and u.

Def 9.1 The number $(u,v) = |u||v|\cos\varphi$ is called the scalar product (dot product) of vectors u and v.

Th 9.3 Let $u, v, w \in \mathbb{R}^3$ ($\mathbb{R}^2$), $a \in \mathbb{R}$. Then

1) $(u+v, w) = (u,v) + (v,w)$, $(au, v) = a(u,v)$
2) $(u,u) = |u|^2 \geq 0$, $|u| = \sqrt{(u,u)}$
3) $(u,u) = 0$ iff $u = 0$
4) $(u,v) = (v,u)$
5) $(u,v) = 0$ iff u, v are orthogonal
6) $(u,v) = a_1 b_1 + a_2 b_2 + a_3 b_3$, if $u = (a_1, a_2, a_3)$, $v = (b_1, b_2, b_3)$
7) $\cos\varphi = \dfrac{(u,v)}{|u||v|}$, where φ is the angle between u, v.

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<u>Proof</u> We only check 6) since other properties are trivial.

We compute

$$|u+v|^2 = |(a_1+b_1, a_2+b_2, a_3+b_3)|^2 =$$

$$= (a_1+b_1)^2 + (a_2+b_2)^2 + (a_3+b_3)^2 =$$

$$= \underbrace{a_1^2 + a_2^2 + a_3^2}_{|u|^2} + \underbrace{b_1^2 + b_2^2 + b_3^2}_{|v|^2} + 2\underbrace{(a_1 b_1 + a_2 b_2 + a_3 b_3)}_{(u,v)}.$$

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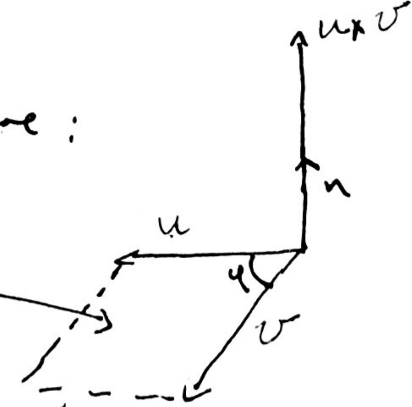
<u>Def 9.2</u> Vectors v_1, v_2, v_3 in $\mathbb{R}^3$ are called an <u>orthonormal basis</u> in $\mathbb{R}^3$ if they are orthogonal and have length 1, that is

$$(v_i, v_j) = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}.$$

3 Vector product on $\mathbb{R}^3$

<u>Def 9.3</u> The vector product (cross product) $u \times v$ is defined as a vector w that is orthogonal to u and v, with a direction given by the right hand rule and the length of the area of the parallelogram that the vectors span.

$$u \times v = \underbrace{|u||v| \sin\varphi}_{\text{area}} \cdot n, \text{ where:}$$



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We note that

$$u \times v = -v \times u.$$

<u>Th 9.4</u> Let $u, v, w \in \mathbb{R}^3$ and $a \in \mathbb{R}$. Then

1) $u \times v = 0$ iff u, v are collinear (parallel)

2) $(u+v) \times w = u \times w + v \times w$

3) $(\alpha u) \times v = \alpha (u \times v)$

4) if i, j, k be an oriented orthonormal basis, i.e they form an orthonormal basis and $i \times j = k$, then

$$u \times v = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

for $u = a_1 i + a_2 j + a_3 k$
$v = b_1 i + b_2 j + b_3 k$

5) $(u, v \times w) = (v, w \times u) = (w, u \times v)$.

6) $u \times (v \times w) = v(u, w) - w(u, v)$

4. Inner product

Let V be a vector space over $\mathbb{F}$.

<u>Def. 9.4</u> An inner product on V is a map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$$

satisfying the following properties

1) Linearity in first slot: $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
$\langle a u, v \rangle = a \langle u, v \rangle$,

for all $u, v, w \in V$ and $a \in \mathbb{F}$;

(7)

2) <u>Positivity</u>: $\langle v, v\rangle \geq 0$ for all $v \in V$;

3) <u>Positive definiteness</u>: $\langle v, v\rangle = 0$ iff $v=0$;

4) <u>Conjugate symmetry</u>: $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$

(if $\mathbb{F}=\mathbb{R}$, then $\langle u,v\rangle = \langle v, u\rangle$)

<u>Remark</u>. $\langle u, a v\rangle = \bar{a}\langle u, v\rangle$

<u>Def 9.5</u> An <u>inner product space</u> is a vector space over $\mathbb{F}$ together with an inner product $\langle\cdot,\cdot\rangle$.

<u>Ex 9.2</u> Let $V = \mathbb{F}^n$, $u = (a_1, \dots, a_n)$, $v = (b_1, \dots, b_n)$

Then $$\langle u, v\rangle = \sum_{i=1}^{n} a_i \bar{b}_i$$

If $\mathbb{F}=\mathbb{R}$, then $\langle u,v\rangle$ is the usual scalar product $\langle u,v\rangle = a_1 b_1 + \dots + a_n b_n$

<u>Ex 9.3</u> $V = \mathbb{F}[z]$ or $V = C([0,1])$

$$\langle f, g\rangle = \int_0^1 f(z)\overline{g(z)}\,dz.$$

<u>Def 9.6</u> A map
$$\|\cdot\| : V \to [0, +\infty)$$
is a <u>norm</u> on V if

1) Positive homogeneity: $\|a v\| = |a|\,\|v\|$ $\forall a \in \mathbb{F}$, $v \in V$;

2) Positive definiteness: $\|v\| = 0$ iff $v=0$;

3) Triangle inequality $\|v + w\| \leq \|v\| + \|w\|$ $\forall v, w \in V$.

If V is an inner product space, we can define

$$\|u\| := \sqrt{\langle u, u\rangle}. \qquad (9.2)$$

It terns out that $\|\cdot\|$ is a norm on V.

Indeed, conditions 1), 2) of Def 9.6 trivially follows from definition of the inner product. We need to check only triangular inequality. First, we prove the following theorem:

<u>Th 9.5 (Cauchy-Schwarz inequality)</u>. Let $\|\cdot\|$ be defined by (9.2). Then for all $u, v \in V$

$$|\langle u, v\rangle| \leq \|u\| \cdot \|v\|.$$

Moreover the equality holds iff u and v are linearly dependent.

<u>Proof</u>. Let $\|v\| \neq 0$. Take $\lambda = \dfrac{\langle u, v\rangle}{\|v\|^2}$

We compute

$$0 \leq \|u - \lambda \cdot v\|^2 = \langle u - \lambda v, u - \lambda v\rangle = \langle u, u\rangle -$$

$$- \langle \lambda v, u\rangle - \langle u, \lambda v\rangle + \langle \lambda v, \lambda v\rangle = \langle u, u\rangle -$$

$$- \lambda \overline{\langle u, v\rangle} - \bar{\lambda}\langle u, v\rangle + \lambda\bar{\lambda}\langle v, v\rangle =$$

$$= \|u\|^2 - \frac{|\langle u, v\rangle|^2}{\|v\|^2} - \frac{|\langle u, v\rangle|^2}{\|v\|^2} + \frac{|\langle u, v\rangle|^2}{\|v\|^2} =$$

$$= \|u\|^2 - \frac{|\langle u, v\rangle|^2}{\|v\|^2}.$$

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