Lecture N8. Eigenvalues and Eigenvectors.

1. Definitions.

In this lecture we are going to answer the question:

Does there exist a basis such that an operator $T: V \to V$ has a diagonal matrix in this basis?

Let V be a finite-dimensional vector space over $\mathbb{F}$.

<u>Def. 8.1</u> A linear operator $T: V \to V$ is called <u>diagonalizable</u> if there exists a basis $v_1, \dots, v_n$ in V such that

$$\mathcal{M}_T = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & & \lambda_n \end{pmatrix} \qquad (8.1)$$

in the basis $v_1, \dots, v_n$

<u>Remark 8.1</u> If $\mathcal{M}_T$ has a form (8.1), then

$$T v_i = \lambda_i v_i \quad \text{for all } i = 1, \dots, n.$$

<u>Def 8.2</u> • A number $\lambda \in \mathbb{F}$ is called an <u>eigenvalue of T</u> if there exists $v \neq 0$ from V such that $Tv = \lambda v$

- The vector v is called an <u>eigenvector of T corresponding to the eigenvalue λ.</u>

<u>Ex 8.1</u> a) $Ov = 0 \quad \forall v \in V.$ (zero map)

Then
- $\lambda = 0$ is an eigenvalue of O
- any vector $v \overset{\neq 0}{\in} V$ is an eigenvector of O corr. to $\lambda = 0$.

b) $Iv = v \quad \forall v \in V$ (identity map)

- $\lambda = 1$ – an eigenvalue of I
- any vector $v \overset{\neq 0}{\in} V$ is an eigenvector of I corr. to $\lambda =$.

c) $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(x,y,z) = (x,y,0)$

- $\lambda = 1$, $v = (1,0,0)$, $v = (0,1,0)$

eigenvalue eigenvectors

- $\lambda = 0$, $v = (0,0,1)$

<u>Def 8.3</u> The set of all eigenvalues of a linear operator $T: V \to V$ is called the <u>spectrum</u> of T. Notation: <u>$\operatorname{Spec} T$</u>

<u>Def 8.4</u> Let $T: V \to V$ be a linear map and λ be an eigenvalue of T

$$V_\lambda = \{ v \in V : Tv = \lambda v \} = \operatorname{Ker}(T - \lambda I)$$

is called an <u>eigenspace</u> of T.

We remark that λ is an eigenvalue of T if $\operatorname{Ker}(T-\lambda I)$ containes non-zero vectors, that is, $T-\lambda I$ is not injective.

The following conditions are equivalent:

- $T-\lambda I$ is not injective ($\dim \operatorname{Ker}(T-\lambda I)>0$)
- $T-\lambda I$ is not invertible
- $T-\lambda I$ is not surjective ($\operatorname{rank}(T-\lambda I)<n$)

2. <u>characteristic polynomial.</u>

Let

$$A = M_T = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ - & - & - & - \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

be a matrix of T is some basis $v_1, \dots, v_n$.

<u>Def 8.5</u> • The matrix

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ - & - & - & - \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{pmatrix} = A - \lambda I$$

is called the <u>characteristic matrix of A</u>

- $\det(A-\lambda I)$ is called the <u>characteristic polynomial</u> of matrix A.

<u>Th 8.1</u> A number λ is an eigenvalue of T iff it is a root of the characteristic polynomial of matrix M_T
(shortly $\lambda \in \operatorname{Spec} T \Leftrightarrow \det(M_T - \lambda I) = 0$.)

<u>Proof.</u> $\lambda \in \operatorname{Spec} T \Leftrightarrow T - \lambda I$ is not injective $\Leftrightarrow$
$\Leftrightarrow T - \lambda I$ is not surjective $\Leftrightarrow$
$\Leftrightarrow \operatorname{rank}(T - \lambda I) < n \Leftrightarrow \det(M_T - \lambda I) = 0$.

∎

<u>Ex 8.2</u> $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T(x,y) = (-y, x)$$

Let write M_T in the standard basis
$e_1 = (1,0)$, $e_2 = (0,1)$

$$Te_1 = (0,1), \quad Te_2 = (-1,0)$$

$$M_T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

To find $\operatorname{Spec} T$, we compute

$$\det\begin{pmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{pmatrix} = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

Since the eqation $\lambda^2 + 1$ has no solutions in $\mathbb{R}$, T has no eigenvalues.

5

<u>Th 8.2</u> Let M_T^e and $M_T^{e'}$ be matrices of a linear operator T in basis $e_1, \dots, e_n$ and $e_1', \dots, e_n'$, respectively. Then

$$\det(M_T^e - \lambda I) = \det(M_T^{e'} - \lambda I).$$

In other words, the characteristic polynomial of the matrix of T does not depend on choice of basis.

<u>Proof.</u> Let $Q = Q_{ee'}$ be a change-of-basis matrix from e to e'. Then, by (7.5)

$$M_T^{e'} = Q^{-1} M_T^e Q$$

So,

$$M_T^{e'} - \lambda I = Q^{-1} M_T^e Q - \lambda I =$$

$$= Q^{-1} M_T^e Q - Q^{-1} \lambda I Q =$$

$$= Q^{-1} (M_T^e - \lambda I) Q.$$

Thus,

$$\det(M_T^{e'} - \lambda I) = \overbrace{\det Q^{-1}}^{=\frac{1}{\det Q}} \cdot \det(M_T^e - \lambda I) \cdot \det Q =$$

$$= \det(M_T^e - \lambda I).$$

<u>Corollary 8.1</u> The characteristic polynomials of similar matrices coinside.

⑥

<u>Th 8.3</u> Let $\lambda_1, \dots, \lambda_m$ be distinct eigenvalues of T with corresponding eigenvectors $v_1, \dots, v_m$. Then $v_1, \dots, v_m$ are linearly independent.

<u>Proof.</u> We assume that $v_1, \dots, v_m$ are linearly dependent. Then there exists $k = 2, \dots, m$ such that $v_1, \dots, v_{k-1}$ are linearly independent and

$$v_k \in \operatorname{span}\{v_1, \dots, v_{k-1}\}.$$

This means that $\exists\, a_1, \dots, a_{k-1} \in \mathbb{F}$ such that

$$v_k = a_1 v_1 + \dots + a_{k-1} v_{k-1} \qquad (8.2)$$

Applying T to both sides, we have

$$\lambda_k v_k = a_1 \lambda_1 v_1 + \dots + a_{k-1} \lambda_{k-1} v_{k-1} \quad (8.3)$$

substracting λ_k times (8.2), we have

$$a_1 (\underbrace{\lambda_1 - \lambda_k}_{\neq 0}) v_1 + \dots + a_{k-1} (\underbrace{\lambda_1 - \lambda_{k-1}}_{\neq 0}) v_{k-1} = 0$$

$\Rightarrow v_1, \dots, v_{k-1}$ - linearly dependent.

This contradict the choise of k. ∎

3. Diagonalization

<u>Th 8.4</u> A linear map $T: V \to V$ is diagonalizable iff there exists a basis $v_1, \dots, v_n$ of V consisting of eigenvectors of T. Moreover, if T is diagonalizable,

and M_T is the matrix of T in basis $v_1, \dots, v_n$, then

$$M_T = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ - & - & - & \cdot \\ 0 & 0 & \dots & \lambda_n \end{pmatrix},$$

where λ_i is the eigenvalue corresponding to v_i $\forall i = 1, \dots, n$.

<u>Ex 8.3</u> $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$, $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

(A map $T : \mathbb{R}^2 \to \mathbb{R}^2$ is identified with A)

$$A v_1 = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2 v_1$$

$$A v_2 = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 5 v_2 .$$

So, v_1, v_2 are eigenvectors which corr. to $\lambda_1 = -2$, $\lambda_2 = 5$. By Th 8.3, v_1, v_2 are independent. So, they form a basis.

By Th. 8.4,

$$M_A = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$$

in basis v_1, v_2.