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Lecture N 7. Change of basis

1. Inverse matrix

Let $A \in \mathbb{F}^{n\times n}$. We recall that:

1) i-j minor of A, denoted by M_{ij}, is the determinant of the matrix obtained by removing the i^{th}-raw and j^{th} column from A.

2) i-j cofactor of A, denoted by A_{ij}, is

$$A_{ij} = (-1)^{i+j} M_{ij}.$$

3) $\det A = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{i=1}^{n} a_{ij} A_{ij}$ - cofactor expansion of det.

Def 7.1 The matrix

$$\operatorname{adj} A = \begin{pmatrix} A_{11} & \dots & A_{n1} \\ - & - & - \\ A_{1n} & \dots & A_{nn} \end{pmatrix}$$

is called the classical adjoint of A.

Note: Here A_{ij} is written in j^{th}-row, i^{th}-column!

Th 7.1 Let $A \in \mathbb{F}^{n\times n}$. The matrix A is invertible iff $\det A \neq 0$. If A is invertible, then

$$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A.$$

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Ex 7.1

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$\det A = -2$

$$\operatorname{adj} A = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}, \quad A^{-1} = \frac{-1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

2. Cramer's rule of solving of system of linear equations

- Consider

$$\begin{cases} a_{11} x_1 + \dots + a_{1n} x_n = b_1 \\ - - - - - - - \\ a_{n1} x_1 + \dots + a_{nn} x_n = b_n \end{cases} \qquad (7.1)$$

and rewrite in the form

$$Ax = b,$$

where $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ - & - & - \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

- By Th 7.1,

$$\det A \cdot x = \operatorname{adj} A \cdot Ax = \operatorname{adj} A \cdot b$$

So, $$\det A \, x_j = \sum_{i=1}^{n} (\operatorname{adj} A)_{ji} b_i = \sum_{i=1}^{n} A_{ij} b_i =$$

$$= \begin{vmatrix} a_{11} \dots b_1 \dots a_{1n} \\ - - - - - \\ a_{n1} \dots b_n \dots a_{nn} \end{vmatrix} = \det B_j$$

j^{th} column is replased by b.

(3) Thus, system of linear equations (7.1) has a unique solution iff $\det A \neq 0$.
Moreover

$$x_j = \frac{\det B_j}{\det A},$$

where B_j is the $n \times n$ matrix obtained from A by replacing the j^{th} column of A by b.

3. Change of basis

Let V be an n-dimensional vector space over $\mathbb{F}$ and let

$$e_1, \dots, e_n$$

be a basis of V.
We recall that then any vector v of V can be uniquelly written as

$$v = a_1 e_1 + \dots + a_n e_n,$$

where $a_1, \dots, a_n$ are some constants from $\mathbb{F}$ and are called the coordinates of v. We will denote them

$$M_v^e = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_e$$

we put index $e = (e_1, \dots, e_n)$ in order to emphasize that it is coordinates on basis $e_1, \dots, e_n$.

(4)

Let $e_1', \ldots, e_n'$ be another basis of V. Then for each $i = 1, \ldots, n$ we can write e_i' in basis $e_1, \ldots, e_n$:

$$e_j' = \sum_{i=1}^{n} \tau_{ij} e_i .$$

<u>Def 7.2</u> The matrix

$$Q_{ee'} = Q = \begin{pmatrix} \tau_{11} & \ldots & \tau_{1n} \\ - & - & - \\ \tau_{n1} & \ldots & \tau_{nn} \end{pmatrix},$$

whose column are the column of the coordinates of the vectors $e_1', \ldots, e_n'$ in basis $e_1, \ldots, e_n$ is called <u>the change-of-basis matrix from basis $e_1, \ldots, e_n$ to basis $e_1', \ldots, e_n'$</u>.

If $e' = (e_1', \ldots, e_n')$, $e = (e_1, \ldots, e_n)$ then

$$e' = e\, Q_{ee'} \qquad (7.2)$$

<u>Th 7.2</u> Let Q be the change-of-basis matrix from basis e to basis e'. Then Q is invertible and Q^{-1} is the change-of-basis matrix from e' to e.

<u>Proof</u> Let Q' be the change-of-basis matrix from e' to e, that is

$$e = e' Q' \qquad (7.3)$$

Then from (7.2) and (7.3)

$$e = e Q Q'.$$

From linear independents of $e_1, \dots, e_n$, we have that

$$Q Q' = I.$$

By Corollary 5.1, Q is invertible and $Q^{-1} = Q'$.

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Now we consider transformation of vector coordinates.

Let $v = a_1 e_1 + \dots + a_n e_n = a_1' e_1' + \dots + a_n' e_n'$,

that is,
$$M_v^e = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}_e, \quad M_v^{e'} = \begin{pmatrix} a_1' \\ a_2' \\ \vdots \\ a_n' \end{pmatrix}.$$

Consequently, we can compute:

$$v = \sum_{j=1}^{n} a_j' \Big(\underbrace{\sum_{i=1}^{n} q_{ij} e_i}_{= e_j'} \Big) = \sum_{i=1}^{n} \underbrace{\Big(\sum_{j=1}^{n} a_j' q_{ij} \Big)}_{= a_i} e_i$$

Thus,
$$a_i = \sum_{j=1}^{n} a_j' q_{ij}, \text{ or in matrix form}$$

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_e = Q_{ee'} \begin{pmatrix} a_1' \\ \vdots \\ a_n' \end{pmatrix}_{e'}.$$

So,
$$\boxed{\begin{pmatrix} a_1' \\ \vdots \\ a_n' \end{pmatrix}_{e'} = Q_{ee'}^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_e = Q_{e'e} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}_e}$$

– transformation of vector coordinates

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Ex 7.2 Consider a three-dim real vector space with basis

$$e_1, e_2, e_3$$

The vectors

$$\begin{cases} e_1' = 5e_1 - e_2 - 2e_3 \\ e_2' = 2e_1 + 3e_2 \\ e_3' = -2e_1 + e_2 + e_3 \end{cases}$$

$$Q_{ee'} = \begin{pmatrix} 5 & 2 & -2 \\ -1 & 3 & 1 \\ -2 & 0 & 1 \end{pmatrix}$$ - change-of-basis matrix from e_1, e_2, e_3 to e_1', e_2', e_3'

$$Q_{ee'}^{-1} = Q_{e'e} = \begin{pmatrix} 3 & -2 & 8 \\ -1 & 1 & -3 \\ 6 & -4 & 17 \end{pmatrix}$$

The vector

$$v = e_1 + 4e_2 - e_3$$

has coordinates in e_1', e_2', e_3':

$$\begin{pmatrix} a_1' \\ a_2' \\ a_3' \end{pmatrix}_{e'} = \begin{pmatrix} 3 & -2 & 8 \\ -1 & 1 & -3 \\ 6 & -4 & 17 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} -13 \\ 6 \\ -27 \end{pmatrix}$$

So

$$v = -13e_1' + 6e_2' - 27e_3'.$$

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4. Relationship between matrices of a linear transformation in different bases.

Let V and W be vector spases over $\mathbb{F}$.

Let also $e = (e_1, \dots, e_n)$ and $e' = (e'_1, \dots, e'_n)$ be bases in V; and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$, $\varepsilon' = (\varepsilon'_1, \dots, \varepsilon'_m)$ be bases in W.

We consider a linear map

$$T : V \to W.$$

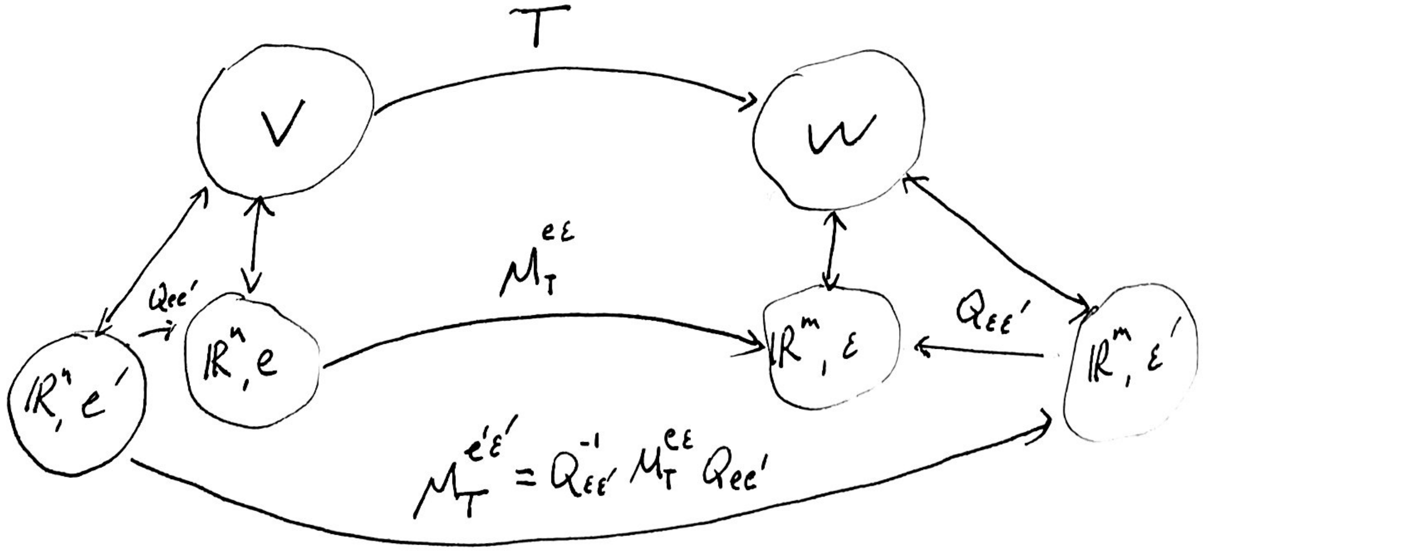
and obtain a relationship between its matrices in different bases.

As before, let $Q_{ee'}$ be a matrix from e to e' and $Q_{\varepsilon\varepsilon'}$ be a change-of-basis matrix from ε to ε', that is,

$$e' = e Q_{ee'}, \qquad \varepsilon' = \varepsilon \cdot Q_{\varepsilon\varepsilon'}. \qquad (7.4)$$

let $M = M_T^{e\varepsilon}$ be a matrix of T in bases e, ε

$M' = M_T^{e'\varepsilon'}$ — // — of T in bases e', ε'



Since the coordinates of Te_j in $\mathcal{E}$ is the j^{th} column of M (see (3.1)), we have that

$$Te = (Te_1, \dots, Te_n) = \mathcal{E} M .$$

Similarly

$$Te' = \mathcal{E}' M' . \qquad (7.5)$$

By (7.4) and (7.5)

$$Te' = T(e \cdot Q_{ee'}) = \mathcal{E}' M' = (\mathcal{E} Q_{\mathcal{E}\mathcal{E}'}) M'$$

So, $$T(e \cdot Q_{ee'}) = (\mathcal{E} Q_{\mathcal{E}\mathcal{E}'}) M' = \mathcal{E} (Q_{\mathcal{E}\mathcal{E}'} M')$$

$$\| $$

$$= (Te) Q_{ee'}$$

$\mathcal{E} M Q_{ee'}$

Consequently,

$$\mathcal{E} M Q_{ee'} = \mathcal{E} Q_{\mathcal{E}\mathcal{E}'} M'$$

By the linear independence of $\mathcal{E}$,

$$M Q_{ee'} = Q_{\mathcal{E}\mathcal{E}'} M' .$$

Consequently,

$$M' = Q_{\mathcal{E}\mathcal{E}'}^{-1} M Q_{ee'}$$

we have obtained

$$M_T^{e'\mathcal{E}'} = Q_{\mathcal{E}\mathcal{E}'}^{-1} M_T^{e\mathcal{E}} Q_{ee'}$$

or

$$M_T^{e'\mathcal{E}'} = Q_{\mathcal{E}'\mathcal{E}} M_T^{e\mathcal{E}} Q_{ee'}$$

Let $W = V$ and $\varepsilon = e$, $\varepsilon' = e'$, then we obtain the formula

$$M_T^{e'e'} = Q_{ee'}^{-1} M_T^{ee} Q_{ee'}. \qquad (7.5)$$

<u>Def 7.3</u> Square matrices and are called <u>similar</u> if there exists a matrix Q such that Q is invertible and

$$A = Q^{-1} B Q$$

<u>Remark 7.1</u> Two matrices ~~A and~~ are similar if and only if they represent one and the same linear map in different bases.

<u>Corollary 7.1</u> Let A, B be similar matrices. Then $\det A = \det B$

<u>Proof</u> Since A, B are similar, there exists an invertible matrix Q, such that

$$A = Q^{-1} B Q.$$

By Theorem 6.5 4)

$$\det A = \det Q^{-1} \cdot \det B \cdot \det Q.$$

Since $\det Q^{-1} = \frac{1}{\det Q}$, because $\det Q^{-1} \cdot \det Q = \det(Q^{-1}Q) = \det I = 1$,

$$\det A = \det B$$

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