N6 Determinant

1. Permutations.

Def. 6.1 A permutation π of n elements is a bijective map from $\{1, \dots, n\}$ to $\{1, \dots, n\}$. We will use the following notation:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ \pi_1 & \pi_2 & \pi_3 & \pi_4 & \dots & \pi_n \end{pmatrix}$$

- The set of all permutations of n elements is denoted by S_n.

Remark 6.1 Permutation is just a reordering of a list of distinct objects.

Ex 6.1 $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 5 & 2 \end{pmatrix}$

- Here $\pi_1 = 3$, $\pi_2 = 1$, $\pi_3 = 4, \dots$

Th. 6.1. The number of permutations in S_n is $n!$

Ex 6.2 $S_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$

So, $|S_2| = 2$, where $|S_2|$ denotes the number of elements in the set S_2.

Sinse each permutation is a map, we can define a composition of permutations. Indeed, let $\pi, \sigma \in S_n$. Define

$$\pi \circ \sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi_{\sigma_1} & \pi_{\sigma_2} & & \pi_{\sigma_n} \end{pmatrix}$$

Ex 6.3

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Def 6.2 Let $\pi \in S_n$. Then an <u>inversion pair</u> (i,j) of π is a pair $i,j \in \{1,\dots,n\}$ for which $i<j$ but $\pi_i > \pi_j$

Ex 6.4 $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $(2,3)$ - inversion pair

$\pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $(1,3), (2,3)$ - inversion pairs

Def 6.3 Let $\pi \in S_n$. Then <u>the sign</u> of π is defined as follows

$$\operatorname{sign} \pi = (-1)^{\text{number of inversion pairs in } \pi} =$$

$$= \begin{cases} +1, & \text{if the number of inversion pairs in } \pi \text{ is even} \\ -1, & \text{— // — is odd.} \end{cases}$$

if $\operatorname{sign} \pi = +1$, π is called <u>even permutation</u>
if $\operatorname{sign} \pi = -1$, π is called an <u>odd permutation</u>

(3) The permutation

$$t_{ij} = \begin{pmatrix} 1 & 2 & \dots & i \dots j \dots n \\ 1 & 2 & \dots & j \dots i \dots n \end{pmatrix}, \quad i<j$$

is called the transposition

<u>Ex 6.5 (HW)</u>

For each $1 \le i < j \le n$ $\quad \operatorname{sign} t_{ij} = -1$.

- Sometimes it is convenient to use a sycle decomposition of permutation π that is an expression of σ as a product of disjoint cycles $(a_1, \dots, a_k)(b_1, \dots, b_m)\dots$, where π sends a_i to a_{i+1}, i.e. $\pi(a_i) = a_{i+1}$, and $\sigma(a_k) = a_1$.

- <u>Ex. 6.6</u> $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1,3)(2) = (1,3)$ (omit sycles of length 1)

$t_{ij} = (i,j)$.

<u>Th 6.2</u> Each permutation can be written as a composition of transpositions, where each sycle $(i_1, \dots, i_k)$ can be written as compositions of $(i_1, i_k), \dots, (i_1, i_2)$

Ex 6.7

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 6 & 4 & 5 \end{pmatrix} = (1,2,3)(4,6,5) =$$

$$= (1,3) \circ (1,2) \circ (4,5) \circ (4,6)$$

$\blacksquare$

Th 6.3 Let $\pi, \sigma \in S_n$. Then

$$\operatorname{sign}(\pi \circ \sigma) = \operatorname{sign} \pi \cdot \operatorname{sign}(\sigma)$$

Remark 6.1 A permutation is even (odd) if a number of transpositions from the decomposition in Th 6.2 is even (odd, respectively).

2. Determinant

Def 6.4 Let $A = (a_{ij}) \in \mathbb{F}^{n \times n}$.

The number

$$\det A := \sum_{\pi \in S_n} \operatorname{sign}(\pi)\, a_{1\pi_1} \cdots a_{n\pi_n}$$

is called the determinant of A.

Ex 6.8 If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then

$$\det A = a_{11} a_{22} - a_{12} a_{21}$$

Ex 6.9 if $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$,

then

$$\det A = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$

$$= a_{11} a_{22} a_{33} + a_{13} a_{21} a_{32} + a_{12} a_{23} a_{31} -$$

$$- a_{13} a_{22} a_{31} - a_{12} a_{21} a_{33} - a_{11} a_{23} a_{32}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

We will also use the notation:

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \det \begin{pmatrix} a_{11} & \dots & a_{1n} \\ - & - & - \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

Th 6.4 (Main properties of determinant)

a) $$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ - & - & - \\ a_{i1}+a_{i1}' & \dots & a_{in}+a_{in}' \\ - & - & - \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ - & - & - \\ a_{i1} & \dots & a_{in} \\ - & - & - \\ a_{n1} & \dots & a_{nn} \end{vmatrix} +$$

$$+ \begin{vmatrix} a_{11} & \dots & a_{1n} \\ - & - & - \\ a_{i1}' & \dots & a_{in}' \\ - & - & - \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ - & - & - \\ c a_{i1} & \dots & c a_{in} \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = c \begin{vmatrix} a_{11} & \dots & a_{1n} \\ - & - & - \\ a_{i1} & \dots & a_{in} \\ - & - & - \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

(determinant is a linear function of each row of the matrix)

b) If A has two the same rows, then

$$\det A = 0$$

c) $\det \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = 1$

<u>Remark</u> There is only one function from $\mathbb{F}^{n \times n}$ to $\mathbb{F}$, which satisfies properties a) - c) of Th 6.4, that is, these properties uniquely dedermine the determinant of matrix.

See [Hoffman, Kunze Linear Algebra] chapter 5.

<u>Th 6.5 (Further properties of determinants)</u>

Let $A = (a_{ij}) \in \mathbb{F}^{n \times n}$.

1) $\det A = \det A^T$, where A^T is the transpose of A

7

2) If B is obtained from A by adding a multiple of one row of A to another (or a multiple of one column to another), then

$$\det B = \det A.$$

3)

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ - & - & - \\ a_{i1} & \dots & a_{in} \\ - & - & - \\ a_{j1} & \dots & a_{jn} \\ - & - & - \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = - \begin{vmatrix} a_{11} & \dots & a_{1n} \\ - & - & - \\ a_{j1} & \dots & a_{jn} \\ - & - & - \\ a_{i1} & \dots & a_{in} \\ - & - & - \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

(The same property holds for interchanging two collumns)

4) $\det AB = \det A \det B$

5)

$$\begin{vmatrix} a_{11} & & \neq 0 \\ & \ddots & \\ 0 & & a_{nn} \end{vmatrix} = a_{11} \dots a_{nn}$$

non zero elements

6)

$$\begin{vmatrix} A & B \\ O & C \end{vmatrix} = |A| \cdot |C|,$$ where

A, B, C are matrices and

O is the zero matrix

3. Computing determinants with cofactor expansions

Def 6.5 For $i,j = 1,\dots,n$ **i-j minor** of A, denoted M_{ij}, is defined to be the determinant of the matrix obtained by removing the i^{th} row and j^{th} column from A.

- **i-j cofactor** of A is defined to be

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Th 6.6 (cofactor expansion) For each i^{th} row (resp. j^{th} column)

$$\det A = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

- $\left(\text{resp. } \det A = \sum_{i=1}^{n} a_{ij} A_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \right).$

Ex 6.10

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 5 \\ 7 & 6 & 3 \end{vmatrix} = (-1)^{2+1} \cdot (-1) \cdot \begin{vmatrix} 2 & 3 \\ 6 & 3 \end{vmatrix} + (-1)^{2+2} \cdot 2 \cdot \begin{vmatrix} 1 & 3 \\ 7 & 3 \end{vmatrix} +$$

$j=2$

$$+ (-1)^{2+3} \cdot 5 \begin{vmatrix} 1 & 2 \\ 7 & 6 \end{vmatrix}$$

(9)

<u>Ex 6.11</u>

$$\begin{vmatrix} 1 & 5 & -1 & 0 \\ -1 & -5 & 2 & 1 \\ 2 & -7 & 2 & 4 \\ -1 & 5 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 1 & 5 & 2 & 1 \\ 4 & 3 & 2 & 4 \\ -3 & -5 & -2 & 1 \end{vmatrix} =$$

$\mathrm{I}+\mathrm{III}$ $\quad$ $\mathrm{II}+5\cdot\mathrm{III}$

$$= (-1)^{1+3}(-1)\begin{vmatrix} 1 & 5 & 1 \\ 4 & 3 & 4 \\ -3 & -5 & 1 \end{vmatrix} = -(3-20-60+9+20-20) = 68$$