N5. Fundamental system of solutions

1. Rank

(1) **Th 5.1** Let $A, B \in \mathbb{F}^{n\times n}$. Then
$$\operatorname{rank}(AB) \le \min(\operatorname{rank} A, \operatorname{rank} B)$$

Proof. We consider two maps

$$T_A : \mathbb{F}^n \to \mathbb{F}^n$$
$$T_B : \mathbb{F}^n \to \mathbb{F}^n$$

defined as follows

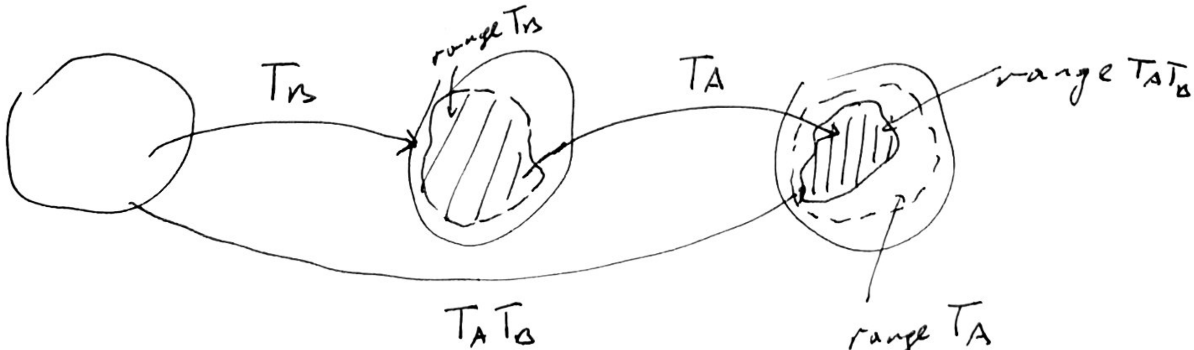
$$T_A x = A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$
$$T_B x = B \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Then A, B are coordinate matrices of T_A, T_B in the standard basis $e_1, \dots, e_n$ of $\mathbb{F}^n$.

By the dimension formula

(5.1) $$\dim V \ge \dim(\operatorname{range} T) = \operatorname{rank} T$$

if $T : V \to W$ is a linear map.



By (5.1.): $\operatorname{rank} T_B \ge \operatorname{rank} T_A T_B$

Moreover $\operatorname{rank} T_A \ge \operatorname{rank} T_A T_B$.

Thus, by Th. 4.2 $\operatorname{rank}(AB) \le \min(\operatorname{rank} A, \operatorname{rank} B)$

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<u>Corollary 5.1</u> A matrix $A \in \mathbb{F}^{n\times n}$ is invertible iff $\exists B \in \mathbb{F}^{n\times n}$ such that $AB = I_n$. In this case $B = A^{-1}$.

<u>Proof.</u> In order to prove the corollary we have tho show that from the existence of $B \in \mathbb{F}^{n\times n}$ such that $AB = I_n$ it follows that A is invertible.

By Th. 4.6

$n = \operatorname{rank} I_n = \operatorname{rank}(AB) \le \operatorname{rank} A$.

Thus, $\operatorname{rank} A = n$. So A is invertible.

Let A^{-1} is its inverse. Then

$$A^{-1} = A^{-1} I_n = A^{-1}(AB) = (A^{-1}A)B = I_n B = B.$$

2 <u>General formula for solutions to a system of linear equations</u>

In this section, we consider a system of linear equations (1.1):

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\ \text{— — — — — —} \\ a_{m1}x_1 + \ldots + a_{mn}x_n = b_m \end{cases} \qquad (5.2)$$

or shortly $Ax = b$.

③ Lemma 5.1 Let x and x' be solutions to (5.2). Then $x-x'$ solves the homogeneous system of linear equations

$$Ay = 0. \qquad (5.3)$$

Proof.

$$A(x-x') = Ax - Ax' = b - b = 0.$$

Corollary 5.2 Let x' be a solution of (5.2). Then the set of all solutions to (5.2) is defined as

$$\{x' + y : y \text{ - solution to } (5.3)\}$$

Let V denote the set of all solutions of homogeneous system of linear equations (5.3). In Ex 2.1 f) we noted that V is a vector space.

Def 5.1 A basis of V is called a fundamental system of solutions to (5.3)

Let $y^1, \dots, y^k$ be a fundamental system of solutions to (4.4). Then The general solution to (5.2) can be given by the formula:

$$x = x' + a_1 y^1 + \dots + a_k y^k, \text{ where}$$

x' is a (partial) solution to (5.2), $a_i \in \mathbb{F}$.

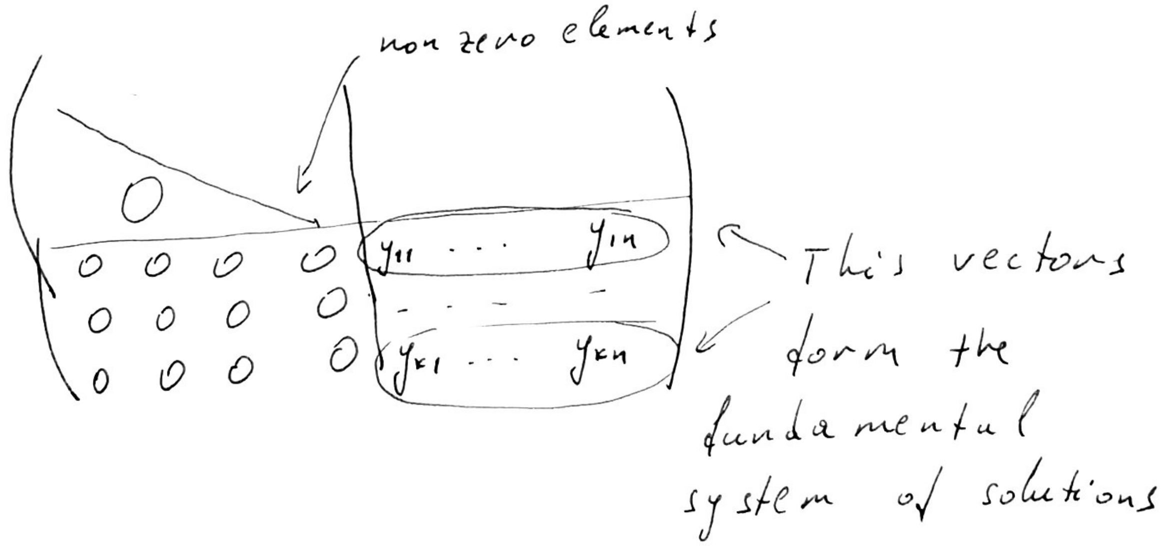
Method of computation of fundamental system of solutions (5.3):

1) Transpose A : $A^T = \begin{pmatrix} a_{11} & \dots & a_{m1} \\ - & - & - \\ a_{1n} & \dots & a_{mn} \end{pmatrix}$

2) Write from the right hand side of A^T the identity matrix I_n :

$$\left(\begin{array}{ccc|ccc} a_{11} & \dots & a_{m1} & 1 & 0 \dots & 0 \\ - & - & - & - & - & - \\ a_{1n} & \dots & a_{mn} & 0 & 0 & 1 \end{array}\right)$$

3) reduce A^T to the row-echelon form doing elementary row transformation (on both matrices)



Ex 5.1 Find fundamental system of solutions to

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_1 - x_2 + x_3 - x_4 = 0 \\ x_2 - x_4 = 0 \end{cases} \qquad (5.4)$$

(5)

$$A = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -2 & -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ II + I \cdot 2 \\ III + I \cdot (-1) \\ \\ \end{array} \sim$$

$$\left(\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right)_{IV + II} \sim \left(\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right)$$

$y^1 = (-1, 0, 1, 0)$
$y^2 = (2, 1, 0, 1)$ – fundamental system of solutions to (5.4)

Method of solving of a system of linear equations (5.2)

The idea is to replace the system (5.2) by the following homogeneous system:

$$\begin{cases} a_{11} x_1 + \ldots + a_{1n} x_n - b_1 x_{n+1} = 0 \\ \text{- - - - - - - - - - -} \\ a_{m1} x_1 + \ldots + a_{mn} x_n - b_m x_{n+1} = 0 \end{cases} \qquad (5.5)$$

1) First we solve (5.5) using the previous method.

⑥

Let $\begin{pmatrix} y_{11} & \dots & y_{1,n+1} \\ - & - & - \\ y_{k1} & \dots & y_{k,n+1} \end{pmatrix}$ be a matrix whose rows are fundamental solutions of (5.5)

2) Reduce it to the form

$$\begin{array}{l} y^1 = \\ \\ y^{k-1} = \\ y^0 = \end{array} \begin{pmatrix} \tilde{y}_{11}, \dots & y_{1,n} & 0 \\ - & - & - \\ y_{k-1,1}, \dots & , y_{k-1,n} & 0 \\ \boxed{y_{k1}, \dots, y_{kn}}, & & 1 \end{pmatrix}$$

fundamental solutions of (5.3)

partial solution of (5.2)

Write the solution as

$$y = y^0 + l_1 y^1 + \dots + l_{k-1} y^{k-1}$$

according to the formula for general solution to (5.2).

Ex 5.2

$$\begin{cases} x_1 + 2x_2 - x_3 - x_4 = 1 \\ x_1 + x_3 + 2x_4 = -1 \\ 2x_1 + 2x_2 + x_4 = 0 \end{cases}$$

(7)

$$\left(\begin{array}{ccc|ccccc} 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ II + I\cdot(-2) \\ III + I \\ IV + I \\ V + I \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|ccccc} 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & -2 & -2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ III + II \\ IV + II \\ V + II \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|ccccc} 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & -2 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ II + IV\cdot 2 \\ \\ \\ \\ \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|ccccc} 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 3 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \boxed{-1 \;\; 1 \;\; 0 \;\; 0} & & & & 1 \end{array}\right)$$

$$(x_1, x_2, x_3, x_4) = (-1, 1, 0, 0) + l_1(-4, 3, 0, 2) +$$
$$+ l_2(-1, 1, 1, 0)$$