Lecture N4. Rank of Matrices

1. Rank of linear maps

<u>Def 4.1</u> Let T be a linear map from V to W, where V, W are vector spaces over $\mathbb{F}$.
The dimension of

$$\text{range } T = \{Tv : v \in V\} =$$
$$= \{w : \exists v \in V \text{ such that } Tv = w\}$$

is called <u>the rank of T</u>, i.e.

$$\text{rank } T = \dim(\text{range } T).$$

<u>Th. 4.1</u> A map $T \in \mathcal{L}(V, W)$ is invertible iff $\dim V = \dim W = \text{rank } T$.

<u>Proof.</u> $\Rightarrow$) If T is invertible, then by dimension formula

$$\dim V = \dim(\ker T) + \underbrace{\dim(\text{range } T)}_{= \text{rank } T}$$

since $\ker T = \{0\}$, because T is injective, and $\text{range } T = W$, because T is surjective, we have $\dim V = \dim W = \text{rank } T$.

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$\Leftarrow$) Let $\dim V = \dim W = \operatorname{rank} T$.

First we note that range T is a vector subspace of W. Since

$$\dim W = \underbrace{\dim(\operatorname{range} T)}_{= \operatorname{rank} T},$$

we have that

$$\operatorname{range} T = W.$$

This implies that T is surjective.

- Next, from the dimension formula it follows that $\dim(\ker T) = 0$.

So, T is injective. Hence, T is invertible.

<u>Def. 4.2</u> Let $A \in \mathbb{F}^{m \times n}$. Then the maximal number of linearly independent columns is called <u>the rank</u> of the matrix A and is denoted by

$$\operatorname{rank} A$$

<u>Th 4.2</u> The rank of a linear map $T \in \mathcal{L}(V, W)$ is equal to the rank of its matrix M_T, that is

$$\operatorname{rank} T = \operatorname{rank} M_T.$$

<u>Corollary 4.1</u> A matrix $A \in \mathbb{F}^{n \times n}$ is invertible iff $\operatorname{rank} A = n$.

Th 4.3 Let $A \in \mathbb{F}^{m \times n}$. The rank of matrix to the maximal number of linearly independent rows.

Th. 4.4 The rank of a matrix is preserved under elementary row and column transformations.

Ex 4.1 Compute $\operatorname{rank}\begin{pmatrix} 1 & 2 & 3 & -1 \\ -1 & 1 & 2 & 1 \\ -1 & 4 & 7 & 1 \end{pmatrix}$

We can make elementary transformations under rows and columns

$$\operatorname{rank}\begin{pmatrix} 1 & 2 & 3 & -1 \\ -1 & 1 & 2 & 1 \\ -1 & 4 & 7 & 1 \end{pmatrix} \underset{\substack{II+I \\ III+I}}{=} \operatorname{rank}\begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 5 & 0 \\ 0 & 6 & 10 & 0 \end{pmatrix}$$

$$= \operatorname{rank}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 6 & 10 & 0 \end{pmatrix} \underset{III+II\cdot(-2)}{=} \operatorname{rank}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \operatorname{rank}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 2.$$

Def 4.3 If $A = (a_{ij})_{i,j=1}^{m,n} \in F^{m \times n}$, then the matrix $A^T = (a_{ji})_{j,i=1}^{n,m} = \begin{pmatrix} a_{11} \dots a_{m1} \\ - - - - - \\ a_{1n} \dots a_{mn} \end{pmatrix}$ is called the transpose matrix of A

Corollary 4.2 $\operatorname{rank} A = \operatorname{rank} A^T$.

Th 4.5 (Rouché - Capelli theorem)
Let A be the matrix of coefficients of a system of linear equations (1.1) and let A' be augmented matrix of (1.1)
Then the system of linear equations (1.1) is consistent iff $\operatorname{rank} A = \operatorname{rank} A'$

Proof In order to proof the theorem we rewrite (1.1) in the following form

$$\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} x_1 + \dots + \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} x_n = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}. \quad (4.1)$$

So, if $\operatorname{rank} A = \operatorname{rank} A'$, then the collumn vector $b = (b_1, \dots, b_m)$ of the matrix A' is a linear combination of other vectors. Indeed, ⑤

Let $A^{(\cdot,j)}$ denote jth column vector of A and $k = \operatorname{rank} A$.

Then, there exists $j_1, \dots, j_k$ such that

$A^{(\cdot,j_1)}, \dots, A^{(\cdot,j_k)}$ are linearly independent and it is a maximal set of linearly independent vectors.

Next, $k = \operatorname{rank} A'$ and $A^{(\cdot,j_1)}, \dots, A^{(\cdot,j_k)}$ are linearli independent, so it is a maximal set of linearly independent vectors. Thus, b can be written as a linear combination of $A^{(\cdot,j_1)}, \dots, A^{(\cdot,j_k)}$.

Hence, (1.1) is consistent.

In the case of consistence of (1.1), (4.1) implies that b is a linear combination of $A^{(\cdot,1)}, \dots, A^{(\cdot,n)}$. Thus,

$$\operatorname{rank} A = \operatorname{rank} A'.$$

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