Lecture N3. Invertible Matrices

1. Matrix of a linear map.

Let V be a vector space with basis $v_1, \dots, v_n$, W be a vector space with basis $w_1, \dots, w_m$ and $T \in L(V, W)$.

Since $w_1, \dots, w_m$ is a basis in W, we can write coordinates of Tv_j in this basis. Let they be

$$M_{Tv_j} = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix},$$

that is

$$Tv_j = a_{1j} w_1 + \dots + a_{mj} w_m$$

We form the following matrix

$$M_T := \begin{pmatrix} a_{11} \dots & a_{1j} & \dots a_{1n} \\ - - & - & - - \\ a_{m1} \dots & a_{mj} & \dots a_{mn} \end{pmatrix} \qquad (3.1)$$

Def 3.1 The matrix M_T is called the matrix of T relative to the pair of bases $v_1, \dots, v_n$ and $w_1, \dots, w_m$.

Th 3.1 Let V be a vector space with basis $v_1, \dots, v_n$, W be a vector space with basis $w_1, \dots, w_m$, $T \in \mathcal{L}(V, W)$ and M_T be defined by (3.1). Then

(i) $M_{Tv} = M_T \cdot M_v$

coordinate matrix of Tv in basis $w_1, \dots, w_m$

coordinate matrix of v in basis $v_1, \dots, v_n$

(ii) Let U be a vector space with basis $u_1, \dots, u_\ell$ and $S \in \mathcal{L}(U, V)$.
Then

$$M_{T \circ S} = M_T \cdot M_S$$

composition of linear maps

(iii) Let $T_1, T_2 \in \mathcal{L}(V, W)$, $c \in \mathbb{F}$.

$$M_{T_1 + T_2} = M_{T_1} + M_{T_2}$$

$$M_{cT} = c\, M_T$$

2 Isomorphism

We recall that

- $T \in \mathcal{L}(V, W)$ is called <u>injective</u> if $v \neq u$ implies $Tv \neq Tu$

  (T is injective iff $\ker T := \{v : Tv = 0\} = \{0\}$)

- T is called surjective if $\operatorname{range} T = W$, where $\operatorname{range} T = \{Tv : v \in V\}$

- T is bijective if T is surjective and injective (T is also called invertible)

We also recall the dimension formula:

(3.2) $\dim V = \dim(\ker T) + \dim(\operatorname{range} T)$

<u>Def 3.2</u> Two vector spaces V and W is called <u>isomorphic</u> if there exists $T \in \mathcal{L}(V, W)$ that is bijective. This map T is called isomorphism of V onto W.

<u>Th. 3.2</u> Every n-dimensional vector space over $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$

<u>Proof</u> Here the isomorphism is given by

$$v \mapsto (a_1, \dots, a_n) = M_v$$

where $(a_1, \dots, a_n)$ are coordinates of v in some fixed basis $v_1, \dots, v_n$.
It is easy to see that this map is bijective (it follows from properties of basis) and

$$M_{v+u} = M_v + M_u$$

$$M_{cv} = c M_v .$$

This shows that the map is linear. $\blacksquare$

<u>Th 3.3</u> Let V be n-dim. vector space over $\mathbb{F}$, W be m-dim vector space over $\mathbb{F}$.
Then $L(V, W)$ is isomorphic to $\mathbb{F}^{m \times n}$

<u>Proof</u> The isomorphism here is given by

$$T \mapsto M_T ,$$

where M_T is the matrix of T relative to some fixed pair of bases $v_1, \dots, v_n$ in V and $w_1, \dots, w_m$ in W.

(5)

Th. 3.4 Let V and W be a vector spaces. Then V and W are isomorphic iff

$$\dim V = \dim W.$$

Proof. $\Rightarrow$) If V and W are isomorphic, then there exists an invertible linear map $T: V \to W$.

By the dimension formula

$$\dim V = \underbrace{\dim(\ker T)}_{\substack{=0 \\ \text{because} \\ T \text{ injective}}} + \dim(\underbrace{\operatorname{range} T}_{\substack{= W \\ \text{because} \\ T \text{ surjective}}})$$

So, $\dim V = \dim W$.

$\Leftarrow$) If $\dim V = \dim W = n$, then we take

$v_1, \dots, v_n$ - basis in V

$w_1, \dots, w_n$ - basis in W

and construct a map such that

$$Tv_i := w_i$$

Note that for any $v \in V$

$$v = a_1 v_1 + \dots + a_n v_n,$$

we have

$$Tv = a_1 Tv_1 + \dots + a_n Tv_n = a_1 w_1 + \dots + a_n w_n.$$

⑥ It is easy to see that
T is a linear invertible map
(check this!)
Hence V and W are isomorphic

3. Invertible linear maps

Let $T: V \to W$ be an invertible linear map. Then for all $w \in W$ there exists a unique $v \in V$ such that

$$Tv = w.$$

We will denote this element v by

$$T^{-1}w := v.$$

We remark, that $T^{-1}: W \to V$ and

$$T \circ T^{-1} w = w,$$
$$T^{-1} T v = v$$

Th. 3.5 Let $T \in \mathcal{L}(V, W)$ be invertible. Then T^{-1} is a linear map from W to V

Since we can identify a linear map with a matrix (see (3.1)), we can introduce a notion of invertible matrix

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Def 3.3 Let A an $n\times n$ (square) matrix over $\mathbb{F}$. The matrix A is called <u>invertible</u> if there exists $B\in\mathbb{F}^{n\times n}$ such that

$$AB = BA = I,$$

where $I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ - & - & - & - \\ 0 & 0 & & 1 \end{pmatrix}$ is the identity matrix.

The matrix B is denoted by A^{-1} and is called <u>an inverse matrix of A</u>.

Th. 3.6 Let $A, B \in \mathbb{F}^{n\times n}$

(i) If A is invertible, then A^{-1} is invertible and $(A^{-1})^{-1} = A$

(ii) If A, B are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(iii) If A is a matrix of a linear map $T\in\mathcal{L}(V,W)$, then

A is invertible iff T is invertible.

Moreover A^{-1} is a matrix of T^{-1}.

Now, we will give a method of computation of an inverse matrix.

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Th 3.7 If A is an $n \times n$-matrix, then the following conditions are equivalent
(i) A is invertible
(ii) A is row-equivalent to te $n \times n$ identity matrix. Moreover if a sequence of elementary row operations reduces A to the identity, then the same sequence of operatoons reduces I to A^{-1}

Ex. 3.1
a)
$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \qquad A^{-1} - ?$$

We write

$$\left(\overbrace{\begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix}}^{A} \middle| \overbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}^{\text{identity matrix}}\right) \overset{I + (-2)II}{\sim}$$

$$\sim \left(\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{\text{identyty}} \middle| \underbrace{\begin{matrix} 1 & -2 \\ 0 & 1 \end{matrix}}_{A^{-1}}\right) \Rightarrow A^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

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<u>Th 3.8</u> For $A \in \mathbb{F}^{n \times n}$ the following are equivalent:

(i) A is invertible

(ii) $Ax = 0$ has only the trivial solution

(iii) $Ax = b$ has a unique solution

$$x = A^{-1} b$$

for every $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$