Lecture N2. Vector spaces

1. Vector spaces and linear maps.

In this section, we recall the main definitions and statements about vector spaces from the previous cource.

- Let V be a set. Let also for each $v, u \in V$, $a \in \mathbb{F}$ there are defined

$$v + u \in V, \quad a \cdot v \in V.$$

Def. 2.1. A vector space over $\mathbb{F}$ is a set V together with the operations addition „+" and multiplication „·"

- satisfying the following conditions:

1) $u + v = v + u, \quad \forall u, v \in V$
2) $(v + u) + w = v + (u + w), \quad \forall u, v, w \in V$
3) $\exists$ an element $0 \in V$ s.t. $0 + v = v \quad \forall v \in V$
4) $\forall v \in V \quad \exists w \in V$ s.t. $s + w = 0$
   (w is denoted by $-v$)
5) $1 \cdot v = v \quad \forall v \in V$;
6) $a(v + u) = av + au$, $(a + b)v = av + bv$
   $\forall v, u \in V$, $a, b \in \mathbb{F}$.

②

Def 2.2 Let $U \subseteq V$. U is called a subspace of V if U is a vector space over $\mathbb{F}$ under the same operations.

Lemma 2.1. $U \subseteq V$ is a subspace of V iff

1) $0 \in U$;
2) $v, u \in U \Rightarrow v + u \in U$;
3) $a \in \mathbb{F}, v \in U \Rightarrow a \cdot v \in U$.

Ex 2.1 a) $\mathbb{F}^n = \{ x = (x_1, \dots, x_n) : x_i \in \mathbb{F}, i = 1, \dots, n \}$

$$x + y = (x_1 + y_1, \dots, x_n + y_n)$$

$$a \cdot x = (a x_1, \dots, a x_n)$$

is a vector space over $\mathbb{F}$.

b) $U = \{ x \in \mathbb{F}^n : x_1 = x_2 = \dots = x_n \} \subseteq \mathbb{F}^n$ is a vector subspace of $\mathbb{F}^n$

c) $\mathbb{F}^{m \times n}$ - the set of $m \times n$-matrices with the usual addition of matrices and multiplication by scalar is also a vector space over $\mathbb{F}$.

d) $\mathbb{F}_n[z]$ - the set of polynomials degree at most n with coefficients from $\mathbb{F}$ is also a vector space over $\mathbb{F}$.

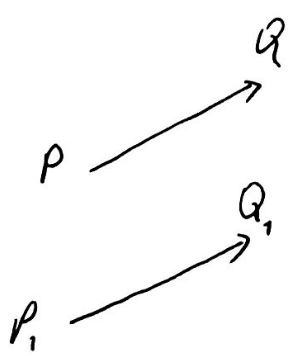
③

e) $\mathbb{F}[z]$ the set of all polynomials with coefficients from $\mathbb{F}$ is a vector space over $\mathbb{F}$.

d) Let
$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n = 0 \\ - - - - - - \\ a_{m1}x_1 + \ldots + a_{mn}x_n = 0 \end{cases} \qquad (2.1)$$
be a homogeneous system of linear equations. Let V be a set of all solutions of (2.1). Then V is a vector subspace of $\mathbb{F}^n$.

<u>The geometric point of view of $\mathbb{R}^3$ ($\mathbb{R}^2$)</u>

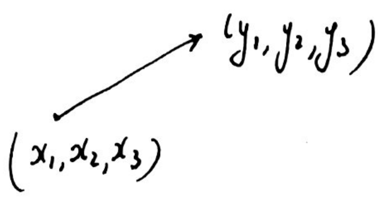
We identify triplets (x_1, x_2, x_3) of real numbers with the points in three-dimensional space. Now we consider an arrow in this three-dim. space as a directed line segment PQ from a point P to a point Q. We will identify arrows with the same length and direction

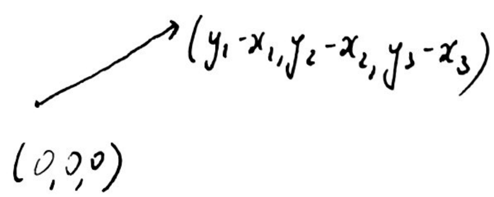


} the same arrows (vectors)

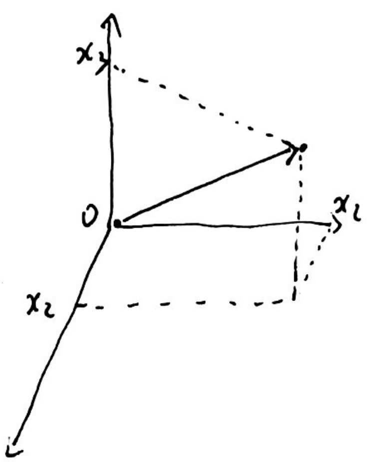
Making this identification, the arrow from the point $P = (x_1, x_2, x_3)$ to $Q = (y_1, y_2, y_3)$ coinsides with the arrow from

$$O = (0,0,0) \quad \text{to} \quad (y_1 - x_1, y_2 - x_2, y_3 - x_3).$$

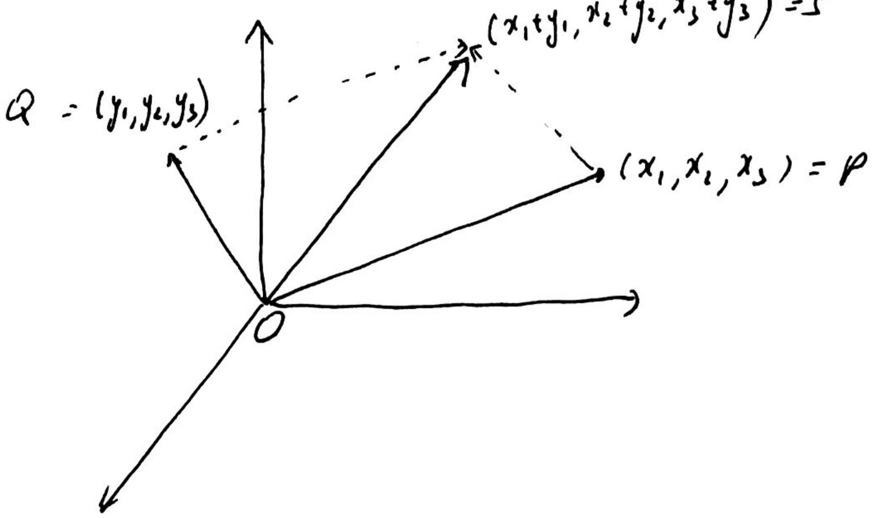




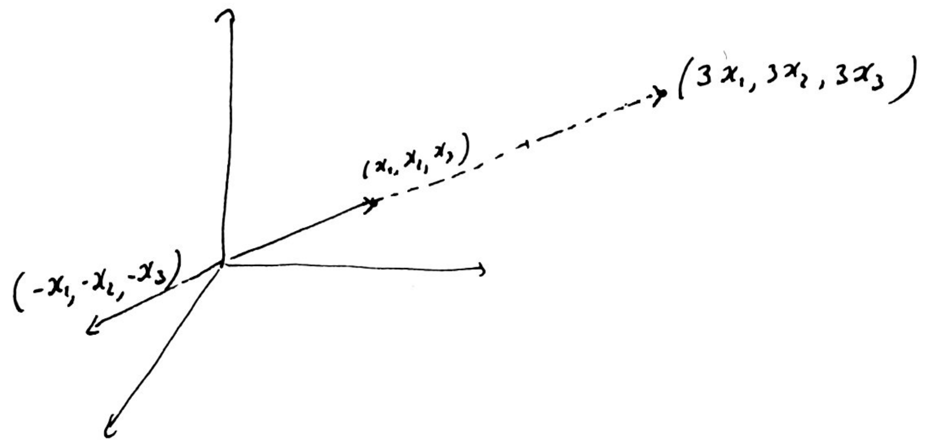
This is the arrow from the origin and it is completely determined by its end point. Thus, we will often identify a triplet (x_1, x_2, x_3) from $\mathbb{R}^3$ with the arrow which starts at $(0,0,0)$ and ends at (x_1, x_2, x_3).



In this case, the definition of sums of two vectors (arrows) can be given geometrically



The sum is defined as the diagonal OS of the parallelogram with edges OP and OQ. Scalar multiplication has a similar geometric interpretation.



2. Bases

Next, we recall the definition of basis of a vector space.

Def 2.3 • $v_1, \dots, v_n \in V$ is called linearly independent if for $a_1, \dots, a_n \in \mathbb{F}$ the equality $a_1 v_1 + \dots + a_n v_n = 0$ implies $a_1 = \dots = a_n = 0$

• The set

$$\text{span}(v_1, \dots, v_n) = \{a_1 v_1 + \dots + a_n v_n : a_i \in \mathbb{F}, i = 1, \dots, n\}$$

is called the linear span of $v_1, \dots, v_n$

⑥ Def 2.4 A list of vectors $v_1, \dots, v_n$ is a basis of V if they are linearly independent and $\operatorname{span}(v_1, \dots, v_n) = V$.

Ex 2.2 a) $e_1 = (1, 0, \dots, 0)$

$e_2 = (0, 1, 0, \dots, 0)$

$- - - - -$

$e_n = (0, \dots, 0, 1)$

- basis in $\mathbb{F}^n$

b) $p_1(z) = 1$

$p_2(z) = z$

$p_3(z) = z^2$

$- - - -$

$p_{n+1}(z) = z^n$

- basis in $\mathbb{F}_n[z]$

c) $v_1 = (1, 1)$

$v_2 = (1, -1)$

- basis in $\mathbb{R}^2$

Th 2.1 Let $v_1, \dots, v_n$ be a basis of a vector space V. Then for each $v \in V$ there exists a unique list of numbers $a_1, \dots, a_n \in \mathbb{F}$ such that

$$v = a_1 v_1 + \dots + a_n v_n$$

<u>Def 2.5</u> The list of numbers $a_1, \dots, a_n$ from Theorem 2.1 we will call <u>coordinates of v relative to the basis $v_1, \dots, v_n$</u> (or in basis $v_1, \dots, v_n$).

Frequently, it will be convenient to use the matrix notation for coordinates of v, that is, we will denote the coordinates of v in the basis $v_1, \dots, v_n$ as

$$M_v := \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

and sometimes $(a_1, a_2, \dots, a_n)$, similarly as elements of $\mathbb{F}^n$.

<u>Ex 2.3</u> We consider

$$\mathbb{F}_1[z] = \{ p(z) = a_0 + a_1 z : a_0, a_1 \in \mathbb{F} \}$$

Let $p_1(z) = 1$

$p_2(z) = z$

Then the coordinates of the polynomial

$p(z) = 2 + z$ in basis p_1, p_2 is

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

but in basis $\tilde{p}_1(z) = 1$, $\tilde{p}_2(z) = z - 1$ it is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, because $p(z) = 3 \cdot 1 + 1 \cdot (z-1) = 3\tilde{p}_1 + 1\tilde{p}_2$.

<u>Def 2.6</u> The <u>dimension of a vector space</u> V is defined as a number of its basis and is denoted by $\dim V$.

<u>Ex 2.4</u> $\dim \mathbb{R}^n = n$, $\dim \mathbb{C}^n = n$,

$\dim \mathbb{F}^{m \times n} = m \cdot n$, $\dim \mathbb{F}_n[z] = n+1$

3. Linear maps

Let V and W be vector spaces over $\mathbb{F}$.

<u>Def 2.7</u> A function $T: V \to W$ is called <u>linear</u> if

1) $T(u+v) = Tu + Tv$, $\forall u, v \in V$,

2) $T(av) = aTv$, $\forall a \in \mathbb{F}, v \in V$.

The set of all linear functions (also called linear transformations or linear maps) is denoted by $\mathcal{L}(V, W)$. If $W = V$, then $\mathcal{L}(V, V) =: \mathcal{L}(V)$.

<u>Remark 2.1</u> The set $\mathcal{L}(V, W)$ is a vector space over $\mathbb{F}$ under the usual operations of additions of functions and multiplication function by a scalar.