Lecture N1. Systems of Linear Equations

1. Definition of a system of Linear Equations

Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

We consider the problem of finding n scalars $x_1, \dots, x_n \in \mathbb{F}$ which satisfies the conditions:

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = b_n \\ \text{-----------------------} \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n = b_m, \end{cases} \qquad (1.1)$$

where b_i, a_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n$ are given numbers from $\mathbb{F}$.

<u>Def 1.1</u> • We call (1.1) a <u>system of m linear equations in n unknowns</u>.

• Any n-type $(x_1, \dots, x_n)$ of elements of $\mathbb{F}$ which satisfies each of the equation of (1) is called a <u>solution</u> of (1)

• If $b_1 = b_2 = \dots = b_n$, then (1) is called <u>homogeneous</u>.

One of the most fundamental technique for finding the solutions of a system of linear equations is the techique of elimination

<u>Ex 1.1</u> $\begin{cases} 2x_1 - x_2 + x_3 = 0, \\ x_1 + 3x_2 + 4x_3 = 0. \end{cases}$ $I + (-2)\cdot II$

Add -2 times the second equation to the first one:

$$\begin{cases} -7x_2 - 7x_3 = 0, \\ x_1 + 3x_2 + 4x_3 = 0 \end{cases} \quad \frac{-1}{7}\cdot I$$

$$\begin{cases} x_2 + x_3 = 0 \\ x_1 + 3x_2 + 4x_3 = 0 \end{cases} \quad II + (-3)\cdot I$$

$$\begin{cases} x_2 + x_3 = 0 \\ x_1 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = -x_3 \\ x_1 = -x_3 \end{cases}$$

Hence the set of solutions consists of all triples $(-a, -a, a)$.

<u>Def 1.2</u> If we multiply jth equation by a scalar $c_j \in \mathbb{F}$, for all $j = 1, \dots, m$, and then add them we get a new equation which is called a <u>linear combination</u> of equations in (1.1)

It is clear that each solution of (1.1) solves the equation which is a linear combination of equations in (1.1).

<u>Def 1.3</u> We will say that two systems of linear equations are <u>equivalent</u> if each eq. in each system is a linear combination of the equations in the other system.

<u>Th. 1.1</u> Equivalent systems of linear equations have the same solutions.

<u>Def. 1.4</u> A system of linear equations is called <u>consistent</u> if it has at least one solution. Otherwise it is called <u>inconsistent</u>.

<u>Ex.1.2</u> The systems:

$$\begin{cases} x_1 + x_2 = 0; \\ 2x_1 + 2x_2 = 1, \end{cases} \qquad \begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 - x_2 + x_3 = 2 \\ 2x_2 + 2x_3 = 3 \end{cases}$$

are inconsistent

The system from Ex. 1.1 is consistent.

2. Matrices and Elementary Row Operations

Def 1.5. Let $m, n \in \mathbb{N}$. A rectangular array of numbers $a_{ij} \in \mathbb{F}$, $i = 1, \dots, m$, $j = 1, \dots, n$

$$A = (a_{ij})_{i,j=1}^{m,n} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ - & - & - & - \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

is called $m \times n$ matrix

- a_{ij} are called entries of A
- we say that i indexes the rows of A and j indexes the columns of A
- we also say that A has size $m \times n$

Def 1.6 The set of all $m \times n$ matrix with entries from $\mathbb{F}$ is denoted by $\mathbb{F}^{m \times n}$

Ex 1.3 $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & i \end{pmatrix} \in \mathbb{C}^{2 \times 3}$, but $A \notin \mathbb{R}^{2 \times 3}$

One can do the following operations on matrices:

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Operations on matrices

1. Multiplication by scalar from $\mathbb{F}$

$$cA = \begin{pmatrix} c\cdot a_{11} & \dots & c\cdot a_{1n} \\ c\cdot a_{m1} & \dots & c\cdot a_{mn} \end{pmatrix}$$

2. Sum of $m\times n$ matrices:

Let $B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ - & - & - \\ b_{m1} & \dots & b_{mn} \end{pmatrix}$

$$A+B = \begin{pmatrix} a_{11}+b_{11} & \dots & a_{1n}+b_{1n} \\ - & - & - \\ a_{m1}+b_{m1} & \dots & a_{mn}+b_{mn} \end{pmatrix}$$

3. Multiplication of $m\times n$ and $n\times l$ matrices

Let $C = \begin{pmatrix} c_{11} & \dots & c_{1l} \\ - & - & - \\ c_{n1} & \dots & c_{nl} \end{pmatrix}$

$$A\cdot C = \begin{pmatrix} d_{11} & \dots & d_{1l} \\ d_{m1} & \dots & d_{ml} \end{pmatrix}$$ $m\times l$ matrix

$$d_{ik} = \sum_{j=1}^{n} a_{ij}\cdot b_{jk}$$

is only defined if the number of columns of A coincide with the number of rows of C !!!

Let A be the matrix with entries which are coefficients of system of linear equations (1.1). We also denote

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Then (1.1) can be rewritten as

$$A \cdot x = b \qquad (1.2)$$

We remark that A is called the <u>matrix of coefficients</u> of the system.

Often it is convenient to consider the <u>augmented matrix</u> of system (1.1) which is defined as follows:

$$A' := \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ - & - & - & \dots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

In the previous section, we made some transformations on the system of linear equations (e.g. replace an equation by a linear combination of equations).

Since we can rewrite the system in form (1.2), it make sence to transfer this operations to matrices and wark with matrix (augmented matrix) of the system instead of the equations.

- We introduce tree <u>elementary row operations</u> on $m \times n$ matrix A:

1) multiplication of one row of A by a non-zero scalar $c \in \mathbb{F}$
2) replacement of the rth row of A by row r plus c times row s ($r \neq s$, $c \in \mathbb{F}$)
3) interchange of two rows of A.

<u>Def 1.7</u> Let $A, B \in \mathbb{F}^{m \times n}$. We say that B is row-equivalent to A if B can be obtained from A by a finite sequence of elementary row operations.

<u>Th 1.2</u> If A and B are row-equivalent augmented matrices of systems of linear equations, then these systems have the same solutions.

3. Row-Reduced Echelon Matrices and Gaussian elimination

- In this section we will consider a method of solving of systems of linear equations.

Let $A \in \mathbb{F}^{m \times n}$ and $A^{(i,\cdot)}$ – ith row vector of A and $A^{(\cdot,j)}$ – jth column vector of A.

<u>Def. 1.8</u> A is in row-echelon form (REF) if the rows of A satisfies:

1) either $A^{(1,\cdot)}$ is the zero vector or the first non-zero entry in $A^{(1,\cdot)}$ (when read from left to right) is a one;
2) for $i = 1, \dots, m$, if $A^{(i,\cdot)} = 0$, then $A^{(i+1,\cdot)} = \dots = A^{(m,\cdot)} = 0$
3) for $i = 2, \dots, m$ if some $A^{(i,\cdot)}$ is not the zero vector, then the first non-zero entry is a one and occurs to the right of the initial one in $A^{(i-1,\cdot)}$

Def. 1.9. The initial leading one is called **pivot**.

Def. 1.10. We say that A is in reduced row-echelon form (RREF) if it is in REF and

4) if $A^{(\cdot;j)}$ contain a pivot, then the pivot is the only non-zero element in $A^{(\cdot;j)}$

Ex 1.4
$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ – REF}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ – RREF}$$

Th 1.3. Every $m \times n$ matrix is row-equivalent to a matrix in reduced row-echelon form

Ex 1.5
$$\begin{pmatrix} 2 & 5 & 3 \\ 1 & 2 & 3 \\ 1 & 0 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix} \begin{matrix} \\ II + (-2)I \\ III + (-1)I \end{matrix} \sim$$

$$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -3 \\ 0 & -2 & 5 \end{pmatrix}_{III + II \cdot 2} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & -1 \end{pmatrix}_{III \cdot (-1)} \sim \underset{\text{REF}}{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\sim\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \leftarrow \text{RREF}$$

In order to solve system (1.1), we can transform the augmented matrix of the system to the RREF, then simply go back to the new system of linear equations, that is equivalent to the old one and then trivially find a solution

<u>Ex 1.6</u>

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ 2x_1 + 5x_2 + 3x_3 = 4 \\ x_1 + 8x_3 = 9 \end{cases}$$

$$\left(\begin{array}{ccc|c}2&5&3&5\\1&2&3&4\\1&0&8&9\end{array}\right) \sim \left(\begin{array}{ccc|c}1&2&3&4\\2&5&3&5\\1&0&8&9\end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c}1&2&3&4\\0&1&-3&-3\\0&-2&5&5\end{array}\right) \sim \left(\begin{array}{ccc|c}1&2&3&4\\0&1&-3&-3\\0&0&+1&+1\end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c}1&2&0&1\\0&1&0&0\\0&0&1&1\end{array}\right) \sim \left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&0\\0&0&1&1\end{array}\right) \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = 1 \end{cases}$$