Retake Solutions

Each of the exercise is 4 points.

1. For which $n \in \mathbb{N}$ the following inequality holds?

$$3^n > 5n + 2. \tag{1}$$

Solution. We first we note that inequality (1) is not true for $n = 1$ ($3^1 < 5 \cdot 1 + 2$) and $n = 2$ ($3^2 < 5 \cdot 2 + 2$). If $n = 3$, then the inequality holds because $3^3 = 27 > 5 \cdot 3 + 2 = 17$. In order to show that inequality (1) is true for all $n \geq 3$, we will use the mathematical induction. Let us assume that (1) holds for $n = k$ for some $k \geq 3$, i.e. $3^k > 5k + 2$, and prove it for $n = k + 1$. So, $3^{n+1} = 3^n \cdot 3 > (5k + 2) \cdot 3 = 15k + 6 = 5(k + 1) + 2 + 10k - 1 > 5(k + 1) + 2$. Hence, the inequality $3^n > 5n + 2$ is true for all $n \geq 3$.

2. Compute the following limit

$$\lim_{n\to\infty} \sqrt[n]{n^5 5^n + n3^n}.$$

Solution. In order to compute the limit we will use the squeeze theorem. For this, we estimate

$$5\left(\sqrt[n]{n}\right)^5 = \sqrt[n]{n^5 5^n} < \sqrt[n]{n^5 5^n + n3^n} < \sqrt[n]{n^5 5^n + n^5 5^n} = 5\sqrt[n]{2n^5} = 5\sqrt[n]{2}\left(\sqrt[n]{n}\right)^5.$$

Since $\lim_{n\to\infty} 5\left(\sqrt[n]{n}\right)^5 = 5 \cdot 1^5 = 5$ and $\lim_{n\to\infty} 5\sqrt[n]{2}\left(\sqrt[n]{n}\right)^5 = 5\lim_{n\to\infty}\sqrt[n]{2} \cdot \lim_{n\to\infty}\left(\sqrt[n]{n}\right)^5 = 5 \cdot 1 \cdot 1^5 = 5$, the squeeze theorem implies that

$$\lim_{n\to\infty} \sqrt[n]{n^5 5^n + n3^n} = 5.$$

3. Show that a sequence $(a_n)_{n\geq 1}$ of real numbers is a Cauchy sequence if and only if $\sup_{n\geq k, m\geq k} |a_n - a_m| \to 0$, $k \to +\infty$.

Solution. Let $(a_n)_{n\geq 1}$ be a Cauchy sequence and let $\varepsilon > 0$ be fixed. By the definition of Cauchy sequence, there exists a number N such that

$$\forall n, m \geq N \quad |a_n - a_m| < \frac{\varepsilon}{2}.$$

Thus, $\sup_{n\geq k, m\geq k} |a_n - a_m| \leq \frac{\varepsilon}{2} < \varepsilon$ for all $k \geq N$. This implies that $\sup_{n\geq k, m\geq k} |a_n - a_m| \to 0$, $k \to +\infty$.

Next, we assume that $\sup_{n\geq k, m\geq k} |a_n - a_m| \to 0$, $k \to +\infty$. Then, by the definition of the convergence, we have that there exists a number N such that

$$\forall k \geq N \quad \sup_{n\geq k, m\geq k} |a_n - a_m| < \varepsilon.$$

In particular, this yields that for all $n, m \geq N$ $|a_n - a_m| < \varepsilon$. So, $(a_n)_{n\geq 1}$ is a Cauchy sequence.

4. For which $a \in \mathbb{R}$ the following function f is differentiable?

$$f(x) = \begin{cases} \frac{\sin x}{e^x - 1}, & x \neq 0, \\ a, & x = 0, \end{cases} \qquad x \in \mathbb{R}.$$

Compute the derivative of f.

Solution. It is clear that the function f is differentiable on $\mathbb{R} \setminus \{0\}$. So, we have to check whether f is differentiable at $x = 0$. For this we first find a for which f is continuous. Only for that a the function could be differentiable. We compute

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{e^x - 1} = \lim_{x \to 0} \frac{x \sin x}{x(e^x - 1)} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{x}{e^x - 1} = 1.$$

So, only for $a = 1$ $\lim_{x \to 0} f(x) = f(0)$. This implies that f is continuous on $\mathbb{R}$ for $a = 1$. Let us check that f is differentiable at $x = 0$ for $a = 1$.

We compute for $a = 1$

$$\begin{aligned} f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\frac{\sin x}{e^x - 1} - 1}{x} = \lim_{x \to 0} \frac{\sin x - e^x + 1}{(e^x - 1)x} \\ &= \lim_{x \to 0} \frac{\not{x} - \frac{x^3}{3!} + o(x^3) - \not{1} - \not{x} - \frac{x^2}{2!} - o(x^2) + \not{1}}{(\not{1} + x + o(x) - \not{1})x} \\ &= \lim_{x \to 0} \frac{\not{x}^2 \left(-\frac{x}{3!} + \frac{o(x^3)}{x^2} - \frac{1}{2!} - \frac{o(x^2)}{x^2} \right)}{\not{x}^2 \left(1 + \frac{o(x)}{x} \right)} = -\frac{1}{2!} = -\frac{1}{2}. \end{aligned}$$

Consequently, $f'(0) = -\frac{1}{2}$ for $a = 1$. If $a \neq 1$, then the function f is not differentiable at $x = 0$ because it is not continuous.

It remains to compute

$$f'(x) = \left(\frac{\sin x}{e^x - 1} \right)' = \frac{\cos x \, (e^x - 1) - e^x \sin x}{(e^x - 1)^2}, \quad x \neq 0.$$

5. Prove that the function $f(x) = x^x$ is increasing on $\left(\frac{1}{e}, \infty\right)$. Is it convex on $\left(\frac{1}{e}, \infty\right)$?

Solution. In order to show that f is increasing, it is enough to show that its derivative is positive. So, we compute

$$f'(x) = (x^x)' = \left(e^{\ln x^x}\right)' = \left(e^{x \ln x}\right)' = e^{x \ln x} \left(\ln x + \frac{x}{x}\right) = x^x (\ln x + 1) > 0$$

for $x \in \left(\frac{1}{e}, \infty\right)$. Hence the function f is strictly increasing on $\left(\frac{1}{e}, \infty\right)$. To check the convexity, we compute the second derivative:

$$f''(x) = (x^x (\ln x + 1))' = (x^x)' (\ln x + 1) + x^x (\ln x + 1)' = x^x (\ln x + 1)^2 + x^x \frac{1}{x} > 0$$

on $\left(\frac{1}{e}, \infty\right)$. Thus, the function f is strictly convex on $\left(\frac{1}{e}, \infty\right)$.

6. Compute the area of the region bounded by the graphs of the following functions $2x = y^2$ and $2y = x^2$.

   Solution. We have to compute the region between two parabolas which intersect each other at points $x = 0$ and $x = 2$ (the points of intersection can be found from the equation $2x = \left(\frac{x^2}{2}\right)^2$). Thus, the area can be computed by the formula

$$\int_0^2 \left(\sqrt{2x} - \frac{x^2}{2}\right) dx = \sqrt{2} \int_0^2 x^{\frac{1}{2}} dx - \frac{1}{2} \int_0^2 x^2 dx = \sqrt{2} \frac{2}{3} x^{\frac{3}{2}} \Big|_0^2 - \frac{1}{6} x^3 \Big|_0^2 = \frac{4}{3}.$$

7. Compute the improper integral $\int_0^\infty |x-1|e^{-x} dx$.

   Solution.

$$\int_0^\infty |x-1|e^{-x} dx = \int_0^1 |x-1|e^{-x} dx + \int_1^\infty |x-1|e^{-x} dx = -\int_0^1 (x-1)e^{-x} dx + \int_1^\infty (x-1)e^{-x} dx.$$

   Let us compute the indefinite integral

$$\int (x-1)e^{-x} dx = -\int (x-1) de^{-x} = -(x-1)e^{-x} + \int e^{-x} d(x-1)$$
$$= -(x-1)e^{-x} + \int e^{-x} dx = -(x-1)e^{-x} - e^{-x} + C = -xe^{-x} + C.$$

   By the fundamental theorem of calculus,

$$\int_0^\infty |x-1|e^{-x} dx = -\left(-xe^{-x}\right)\Big|_0^1 + \left(-xe^{-x}\right)\Big|_1^\infty = e^{-1} - 0 + 0 + e^{-1} = 2e^{-1} = \frac{2}{e}.$$

8. Does the following series converges?

$$\sum_{n=1}^\infty \frac{1}{n\left(\ln^2 n + 1\right)}.$$

   Solution. Since the sequence $\frac{1}{n\left(\ln^2 n+1\right)}$ decreases, we can use the integral criterion. According to that criterion, the convergence of the series $\sum_{n=1}^\infty \frac{1}{n\left(\ln^2 n+1\right)}$ is equivalent to the convergence of the improper integral

$$\int_1^\infty \frac{dx}{x\left(\ln^2 x + 1\right)} = \int_1^\infty \frac{d\ln x}{\ln^2 x + 1} \overset{y=\ln x}{=} \int_0^\infty \frac{dy}{y^2+1} = \arctan y \Big|_0^\infty = \frac{\pi}{2} < \infty.$$

   Hence, the series $\sum_{n=1}^\infty \frac{1}{n\left(\ln^2 n+1\right)}$ converges.

9. Write the following complex numbers in algebraic form: $\frac{i}{(1-i)^2}$, $(1-\sqrt{3}i)^{15}$.

   Solution.

$$\frac{i}{(1-i)^2} = \frac{i}{1-2i+i^2} = \frac{i}{1-2i-1} = \frac{i}{-2i} = -\frac{1}{2}.$$

In order to compute $(1-\sqrt{3}i)^{15}$, we will use de Moivre's formula. For this, we need to rewrite the complex number $1-\sqrt{3}i$ in polar form. So, the absolute value r of $1-\sqrt{3}i$ is given by the formula $r=\sqrt{1^2+\left(\sqrt{3}\right)^2}=2$. The argument θ of $1-\sqrt{3}i$ can be found from the equalities $\cos\theta=\frac{1}{2}$ and $\sin\theta=\frac{\sqrt{3}}{2}$. Hence, $\theta=-\frac{\pi}{3}$. Consequently, we can compute

$$\begin{aligned}(1-\sqrt{3}i)^{15}&=\left(2\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)\right)^{15}=2^{15}\left(\cos\left(-\frac{15\pi}{3}\right)+i\sin\left(-\frac{15\pi}{3}\right)\right)\\&=2^{15}\left(\cos(-5\pi)+i\sin(-5\pi)\right)=-2^{15}.\end{aligned}$$

10. Let $\mathbb{R}_4[z]$ denotes the vector space of all polynomials of degree at most 4 with coefficients in $\mathbb{R}$ and let the linear operator $T:\mathbb{R}_4[z]\to\mathbb{R}_4[z]$ is defined as follows $(T\mathbf{p})(z)=\mathbf{p}''(z)$ ($T\mathbf{p}$ is the second order derivative of polynomial $\mathbf{p}$). Identify $\ker T$ and $\operatorname{range} T$. Find a subspace W of $\mathbb{R}_4[z]$ such that $\ker T\oplus W=\mathbb{R}_4[z]$.

    Solution. We take $\mathbf{p}\in\mathbb{R}_n[z]$. Then $\mathbf{p}$ can be written as $\mathbf{p}(z)=a_4z^4+a_3z^3+a_2z^2+a_1z+a_0$, where $a_i\in\mathbb{R}$, $i=0,\dots,4$. So, by the definition of T,

$$(T\mathbf{p})(z)=12a_4z^2+6a_3z+2a_2\in\mathbb{R}_2[z]. \tag{2}$$

This implies that $T\mathbf{p}=\mathbf{0}$ if and only if $a_2=a_3=a_4=0$. Hence,

$$\ker T=\{\mathbf{p}\in\mathbb{R}_4[z]:\ T\mathbf{p}=\mathbf{0}\}=\{\mathbf{p}(z)=a_1z+a_0:\ a_0,a_1\in\mathbb{R}\}=\mathbb{R}_1[z].$$

Next, we compute

$$\operatorname{range} T=\{\mathbf{q}\in\mathbb{R}_4[z]:\ \exists\mathbf{p}\in\mathbb{R}_4[z]\ \text{ such that }\ T\mathbf{p}=\mathbf{q}\}=\mathbb{R}_2[z].$$

Indeed, if $\mathbf{q}(z)=b_2z^2+b_1z+b_0\in\mathbb{R}_2[z]$, then for $\mathbf{p}(z)=\frac{b_2}{12}z^4+\frac{b_1}{6}z^3+\frac{b_1}{2}z^2$ we trivially have $T\mathbf{p}=\mathbf{q}$. So, $\operatorname{range} T\supset\mathbb{R}_2[z]$. Moreover, equality (2) implies $\operatorname{range} T\subset\mathbb{R}_2[z]$.

In order to find a vector subspace W of $\mathbb{R}_4[z]$ such that $\ker T\oplus W=\mathbb{R}_4[z]$, we recall that it should be a vector subspace such that $\ker T+W=\mathbb{R}_4[z]$ and $\ker T\cap W=\{\mathbf{0}\}$. We set

$$W=\left\{\mathbf{q}(z)=b_4z^4+b_3z^3+b_2z^2:\ b_2,b_3,b_4\in\mathbb{R}\right\}.$$

It is easily to see that W is a vector subspace of $\mathbb{R}_4[z]$ and $\ker T+W=\mathbb{R}_2[z]+W=\mathbb{R}_4[z]$. Moreover, only zero polynomial belongs to both $\ker T=\mathbb{R}_2[z]$ and W. Hence, $\ker T\oplus W=\mathbb{R}_4[z]$.