An example of problem sheet for the exam

Each of the exercise is 4 points.

Solutions will be discussed during the lecture on Wednesday February 6.

1. Prove that all numbers of the form $5^n - 4n - 1$, $n \in \mathbb{N}$ are divisible by 16.
2. Show that $a_n \to 0,\ n \to \infty$ if and only if $\sup\{|a_k| :\ k \geq n\} \to 0,\ n \to \infty$.
3. For which $a, b \in \mathbb{R}$ is the following function continuous on $\mathbb{R}$? For which $a, b \in \mathbb{R}$ is it differentiable on $\mathbb{R}$?
$$f(x) = \begin{cases} ax + b, & \text{if } x \leq 0, \\ 2^x, & \text{if } x > 0. \end{cases}$$
4. Compute the limit $\lim\limits_{x \to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$.
5. Prove that $1 - \frac{x^2}{2} \leq \cos x$ for all $x \geq 0$.
6. Compute the first and the second derivatives of the function $f(x) = \arctan(x^2)$, $x \in \mathbb{R}$.
7. Compute the integral $\int_0^{\sqrt{\pi}} x^3 \sin(x^2) dx$.
8. For which $\alpha > 0$ the integral $\int_1^{+\infty} \frac{\sin x}{x^\alpha} dx$ converges?
9. Investigate the absolute and conditional convergence of the series $\sum\limits_{n=1}^{\infty} (-1)^n \frac{n!}{n^{2n}}$.
10. Prove that $(\sqrt{3} + i)^{120}$ is a real number.
11. Let $\mathbb{R}_n[z]$ denote the real vector space of polynomials of degree at most n with real coefficients. Let $p_0, p_1, \ldots, p_n \in \mathbb{R}_n[z]$ satisfy $p_j(0) = 0$ for all $j = 0, 1, \ldots, n$. Prove that $p_0, p_1, \ldots, p_n$ must be a linearly dependent in $\mathbb{R}_n[z]$.
12. Show that the linear map $T : \mathbb{R}^4 \to \mathbb{R}^2$ is surjective if
$$\ker T = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 :\ x_1 = x_2,\ x_3 = x_4\}.$$
13. Give the definition of the upper and lower limits of a sequence $(a_n)_{n \geq 1}$.
14. Give the definition of a rectifiable curve and write the formula for computation of the length of a rectifiable curve.