9 Lecture 9 – Properties of Continuous Functions

9.1 Boundedness of Continuous Functions and Intermediate Value Theorem

For more details see [1, Section 3.18].

Let $-\infty < a < b < +\infty$ be fixes.

Theorem 9.1 (1st Weierstrass theorem). *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. Then f is bounded on $[a,b]$.*

Proof. We assume that f is unbounded on $[a,b]$. Then for each $n \in \mathbb{N}$ there exists $x_n \in [a,b]$ such that $|f(x_n)| \geq n$. Since the sequence $(x_n)_{n\geq 1}$ is bounded (each x_n belongs to the interval $[a,b]$), it has a convergent subsequence $(x_{n_k})_{k\geq 1}$, by the Bolzano-Weierstrass theorem (see Theorem 4.6). So, let $x_{n_k} \to x_\infty$, $k \to \infty$. Using the inequalities $a \leq x_{n_k} \leq b$ for all $k \geq 1$ and Theorem 3.6, we have that $a \leq x_\infty \leq b$. Since the function f is continuous on $[a,b]$, we have that $f(x_{n_k}) \to f(x_\infty)$, $k \to \infty$. But this is impossible because $|f(x_{n_k})| \geq n_k \to +\infty$, $k \to \infty$. So, the function f must be bounded. □

Example 9.1. If $f : (a,b] \to \mathbb{R}$ is a continuous function on $(a,b]$, then the function could be unbounded. Indeed, we set $(a,b] = (0,1]$ and $f(x) = \frac{1}{x}$, $x \in (0,1]$. Then $f \in \mathrm{C}((0,1])$ but $f(x) \to +\infty$, $x \to 0+$.

Corollary 9.1. *Let $f : [a,+\infty) \to \mathbb{R}$ be a continuous function on $[a,+\infty)$ and $f(x) \to p \in \mathbb{R}$, $x \to +\infty$. Then f is bounded on $[a,+\infty)$.*

Proof. By Theorem 7.1 (iii), for $\varepsilon := 1$ there exists $D > a$ such that $|f(x) - p| < \varepsilon = 1$ for all $x \geq D$. Hence $p - 1 < f(x) < p + 1$ for all $x \geq D$, which implies the boundedness of f on $[D,+\infty)$. Next, since the function is continuous on the interval $[a,D]$, we can apply the 1st Weierstrass theorem. Consequently, f is also bounded on $[a,D]$. Hence the function f is bounded on $[a,+\infty)$. □

Exercise 9.1. Prove that the function $f(x) = \left(1 + \frac{1}{x}\right)^x$, $x > 0$, is bounded on $(0,+\infty)$.

(*Hint:* Theorem 9.1 as well as Corollary 9.1 can not be applied to the interval $(0,+\infty)$, since the the point a does not belong to the interval. First find the limits of f as $x \to 0+$ and $x \to +\infty$) and then use the argument from Corollary 9.1.)

Theorem 9.2 (2nd Weierstrass theorem). *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. Then f assumes its minimum and maximum values on $[a,b]$, that is, there exist x_* and x^* in $[a,b]$ such that $f(x_*) \leq f(x) \leq f(x^*)$ for all $x \in [a,b]$.*

Proof. We will prove the existence of x^*. The proof is similar for x_*. By the 1st Weierstrass theorem, the function f is bounded on $[a,b]$, that implies that the set $f([a,b]) = \{f(x) : x \in [a,b]\}$ is bounded. So, we set $p := \sup f([a,b]) = \sup_{x\in[a,b]} f(x)$, which exists, by Theorem 2.2 (i). According to Theorem 2.1 (i), for each $n \in \mathbb{N}$ there exists $x_n \in [a,b]$ such that $p - \frac{1}{n} < f(x_n) \leq p$. We apply the Bolzano-Weierstrass theorem (see Theorem 4.6) to the sequence $(x_n)_{n\geq 1}$. Consequently, there exists a convergent subsequence $(x_{n_k})_{k\geq 1}$. We denote its limit by x^*. So, $x_{n_k} \to x^*$, $k \to \infty$. Since $f \in \mathrm{C}([a,b])$, we have that $f(x_{n_k}) \to f(x^*)$, $k \to \infty$. Moreover,

$$p - \frac{1}{n_k} < f(x_{n_k}) \leq p$$

for all $k \geq 1$. Hence, $f(x_{n_k}) \to p$, $k \to \infty$, by the Squeeze theorem (see Theorem 3.7). It implies that $f(x^*) = p$. Consequently, $f(x^*) = \sup_{x\in[a,b]} f(x) = \max_{x\in[a,b]} f(x)$, that is, $f(x) \leq f(x^*)$ for all $x \in [a,b]$. □

Exercise 9.2. Prove the existence of the point x_* in the 2nd Weierstrass theorem.

Exercise 9.3. Let $f : [0, +\infty) \to \mathbb{R}$ be a continuous function on $[0, +\infty)$ and $f(x) \to +\infty$, $x \to +\infty$. Show that there exists $x_* \in [0, +\infty)$ such that $f(x_*) = \inf_{x\in[0,+\infty)} f(x) = \min_{x\in[0,+\infty)} f(x)$.

Theorem 9.3 (Intermediate value theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Then for any real number y_0 between $f(a)$ and $f(b)$, i.e. $f(a) \le y_0 \le f(b)$ or $f(b) \le y_0 \le f(a)$, there exists x_0 from $[a, b]$ such that $f(x_0) = y_0$.*

Proof. If $y_0 = f(a)$ or $y_0 = f(b)$, then x_0 equals a or b, respectively. Now we assume that $f(a) < y_0 < f(b)$. The case $f(b) < y_0 < f(a)$ is similar. We set $M := \{x \in [a, b] : f(x) \le y_0\}$, which is non empty set because $a \in M$. Moreover, it is bounded as a subset of the interval $[a, b]$. Consequently, there exists $\sup M =: x_0$.

We are going to show that $f(x_0) = y_0$. According to Theorem 2.1 (i), for each $n \in \mathbb{N}$ there exists $x_n \in M$ such that $x_0 - \frac{1}{n} < x_n \le x_0$. Thus, $x_n \to x_0$, $n \to \infty$, by the Squeeze theorem (see Theorem 3.7). Since $x_n \in M$, we have that $f(x_n) \le y_0$ for all $n \ge 1$. Moreover, $f(x_n) \to f(x_0)$, $n \to \infty$ due to the continuity of f. Thus, using Theorem 3.6, we obtain $f(x_0) \le y_0$.

Next, for every $x > x_0$ we have that $x \notin M$, since x_0 is the supremum of M. It implies that $f(x) > y_0$. Consequently, $y_0 \le \lim_{x\to x_0+} f(x) = \lim_{x\to x_0} f(x) = f(x_0)$. Here we have also used the continuity of f and Theorem 7.8. Thus, $y_0 = f(x_0)$. □

Exercise 9.4. Prove that the function $P(x) = x^3 + 2x^2 - 1$, $x \in \mathbb{R}$, has at least one root, that is, there exists $x_0 \in \mathbb{R}$ such that $P(x_0) = 0$.

Corollary 9.2. *Let $f, g \in \mathrm{C}([a, b])$ and $f(a) \le g(a)$, $f(b) \ge g(b)$. Then there exists $x_0 \in [a, b]$ such that $f(x_0) = g(x_0)$.*

Proof. We note that the function $h(x) := f(x) - g(x)$, $x \in [a, b]$, is continuous on $[a, b]$ and satisfies $h(a) \le 0 \le h(b)$. So, taking $y_0 := 0$ and applying the intermediate value theorem, we obtain that there exists $x_0 \in [a, b]$ such that $h(x_0) = f(x_0) - g(x_0) = 0$. □

Example 9.2. Let $g : [0, 1] \to [0, 1]$ be a continuous function on $[0, 1]$. Then there exists $x_0 \in [0, 1]$ such that $g(x_0) = x_0$.

To prove the existence of x_0, we take $f(x) = x$, $x \in [0, 1]$, and note that f is continuous on $[0, 1]$ and $f(0) = 0 \le g(0)$, $f(1) = 1 \ge g(1)$. Thus, by Corollary 9.2, there exists $x_0 \in [0, 1]$ such that $g(x_0) = f(x_0) = x_0$.

Corollary 9.3. *Let $f \in \mathrm{C}([a, b])$. Then its range $f([a, b]) = \{f(x) : x \in [a, b]\}$ is an interval.*

Proof. By the 2nd Weierstrass theorem (see Theorem 9.2), there exists $x_*, x^* \in [a, b]$ such that $f(x_*) \le f(x) \le f(x^*)$ for all $x \in [a, b]$. Consequently, $f([a, b]) \subset [f(x_*), f(x^*)]$. Next, due to the intermediate value theorem, for each $y_0 \in [f(x_*), f(x^*)]$ there exists $x_0 \in [a, b]$ such that $f(x_0) = y_0$. It implies that $y_0 \in f([a, b])$ and, consequently, $[f(x_*), f(x^*)] \subset f([a, b])$. Hence, $f([a, b]) = [f(x_*), f(x^*)]$. □

Exercise 9.5. Let $f : [a, b] \to \mathbb{R}$ strictly increase on $[a, b]$ and for each $y_0 \in [f(a), f(b)]$ there exist $x_0 \in [a, b]$ such that $f(x_0) = y_0$. Prove that $f \in \mathrm{C}([a, b])$.

Exercise 9.6. Let $f, g : [0, 1] \to [0, 1]$ be continuous and f be a surjection. Prove that there exists $x_0 \in [0, 1]$ such that $f(x_0) = g(x_0)$.

9.2 Uniformly Continuous Functions

For more details see [1, Section 3.19].

Let A be a subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$. We recall that f is continuous at point x_0 provided $\forall\varepsilon > 0$ $\exists\delta > 0$ such that for each $x \in A$ the inequality $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$ (see Definition 8.3). The choice of δ depends on ε and the point x_0. It turns out to be very useful to know when the δ can be chosen to depend only on ε. Such functions are said to be uniformly continuous on A.

Definition 9.1. A function $f : A \to \mathbb{R}$ is said to be **uniformly continuous on** A, if

$$\forall\varepsilon > 0\ \exists\delta > 0\ \forall x', x'' \in A,\ |x' - x''| < \delta :\ |f(x') - f(x'')| < \varepsilon.$$

Remark 9.1. Any uniformly continuous function on A is continuous on A. The converse statement is not true, see Example 9.5 below.

Example 9.3. The function $f(x) = x$, $x \in \mathbb{R}$, is uniformly continuous on $\mathbb{R}$, since for each $\varepsilon > 0$ we can take $\delta := \varepsilon$. Then for all $x', x'' \in \mathbb{R}$ such that $|x' - x''| < \delta$ we have $|f(x') - f(x'')| = |x' - x''| < \delta = \varepsilon$.

Example 9.4. The functions sin and cos are uniformly continuous on $\mathbb{R}$.

The function sin is uniformly continuous on $\mathbb{R}$, since for each $\varepsilon > 0$ we can take $\delta := \varepsilon > 0$. Then for all $x', x'' \in \mathbb{R}$ such that $|x' - x''| < \delta$ we have

$$|\sin x' - \sin x''| = 2\left|\cos\frac{x' + x''}{2}\right| \cdot \left|\sin\frac{x' - x''}{2}\right| \leq 2 \cdot 1 \cdot \frac{|x' - x''|}{2} = |x' - x''| < \delta = \varepsilon,$$

where we have also used Remark 6.3 for the estimation of $\left|\sin\frac{x'-x''}{2}\right|$.

Example 9.5. The function $f(x) = \frac{1}{x}$, $x \in (0, 1]$, is not uniformly continuous on $(0, 1]$.

Indeed, for $\varepsilon := 1$ we have that for all $\delta > 0$ we can take $x' := \frac{1}{n}$ and $x'' := \frac{1}{n+1}$ from $(0, 1]$ such that $|x' - x''| < \delta$ and $\left|\frac{1}{x'} - \frac{1}{x''}\right| = |n - (n + 1)| = 1 = \varepsilon$, where $n \in \mathbb{N}$ and $n > \frac{1}{\delta}$.

Exercise 9.7. Prove that the following functions are uniformly continuous on their domains:
a) $f(x) = \ln x$, $x \in [1, +\infty)$; b) $f(x) = \sqrt{x}$, $x \in [0, +\infty)$; c) $f(x) = x \sin\frac{1}{x}$, $x \in (0, +\infty)$.

Exercise 9.8. Prove that the following functions are not uniformly continuous on their domains:
a) $f(x) = \ln x$, $x \in (0, 1]$; b) $f(x) = \sin(x^2)$, $x \in [0, +\infty)$; c) $f(x) = x \sin x$, $x \in [0, +\infty)$.

Theorem 9.4 (Heine-Cantor theorem)**.** *Let a function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then f is uniformly continuous on $[a, b]$.*

Proof. Be assume that f is not uniformly continuous on $[a, b]$. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists x' and x'' from $[a, b]$ such that $|x' - x''| < \delta$ and $|f(x') - f(x'')| \geq \varepsilon$. So, for each $n \in \mathbb{N}$ taking $\delta := \frac{1}{n}$, we can find x'_n and x''_n from $[a, b]$ such that $|x'_n - x''_n| < \delta = \frac{1}{n}$ and $|f(x'_n) - f(x''_n)| \geq \varepsilon$.

We will consider the obtained sequences $(x'_n)_{n\geq1}$ and $(x''_n)_{n\geq1}$. By the Bolzano-Weierstrass theorem (see Theorem 4.6), there exists a subsequence $(x'_{n_k})_{n\geq1}$ of $(x'_n)_{n\geq1}$ which converges to some real number $x_\infty \in [a, b]$. Since $|x'_{n_k} - x''_{n_k}| < \frac{1}{n_k}$ for all $k \geq 1$, we have $x''_{n_k} \to x_\infty$. By the continuity of f, $f(x'_{n_k}) \to f(x_\infty)$, $k \to \infty$, and $f(x''_{n_k}) \to f(x_\infty)$, $k \to \infty$. But $|f(x'_{n_k}) - f(x''_{n_k})| \geq \varepsilon > 0$, for all $k \geq 1$, that contradict our assumption. □

References

[1] K.A. Ross. *Elementary Analysis: The Theory of Calculus.* Undergraduate Texts in Mathematics. Springer New York, 2013.