8 Lecture 8 – Continuous Functions

8.1 Definitions and Examples

Let $A \subset \mathbb{R}$, $a \in A$ be a limit point of A and $f : A \to \mathbb{R}$.

Definition 8.1. A function f is said to be **continuous at** a, if $\lim_{x\to a} f(x) = f(a)$, i.e. $\lim_{x\to a} f(x) = f\left(\lim_{x\to a} x\right)$.

By the definition of limit of function (see Definition 6.6) and Theorem 7.1, the following two definitions are equivalent to Definition 8.1.

Definition 8.2. A function f is said to be **continuous at** a, if for each sequence $(x_n)_{n\geq 1}$ such that 1) $x_n \in A$ for all $n \geq 1$; 2) $x_n \to a$, $n \to \infty$, it follows that $f(x_n) \to f(a)$, $n \to \infty$.

Definition 8.3. A function f is said to be **continuous at** a, if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in A \cap B(a, \delta) : \ |f(x) - f(a)| < \varepsilon.$$

Now we want to introduce the left and right continuity. For this we assume that $a \in A$ satisfies (7) (resp., (8)).

Definition 8.4. A function f is said to be **left continuous** (resp. **right continuous**), if $f(a-) = f(a)$ (resp. $f(a+) = f(a)$).

Remark 8.1.
1. If $(a - \gamma, a] \subset A$ for some $\gamma > 0$, then f is left continuous iff

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in (a - \delta, a] : \ |f(x) - f(a)| < \varepsilon.$$

   This immediately follows from Theorem 7.6.

2. If $[a, a + \gamma) \subset A$ for some $\gamma > 0$, then f is right continuous iff

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in [a, a + \delta) : \ |f(x) - f(a)| < \varepsilon.$$

   This follows from Theorem 7.7.

Remark 8.2. Let a satisfy properties (7) and (8). Then, by Theorem 7.8, a function f is continuous at the point a iff f is left and right continuous at a.

For convenience we will suppose that every function is continuous at each isolated point, points from A which are not its limit points.

Definition 8.5. A function $f : A \to \mathbb{R}$ is called **continuous** on the set A, if it is continuous at each point of A. We will often use the notation $f \in \mathrm{C}(A)$.

Theorem 8.1. *Let functions $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be continuous at $a \in A$. Then*

a) for each real number c the function $c \cdot f$ is continuous at the point a;

b) the function $f + g$ is continuous at the point a;

c) the function $f \cdot g$ is continuous at the point a;

d) the function $\frac{f}{g}$ is continuous at the point a, if additionally $g(a) \neq 0$.

In the theorem, the functions $c \cdot f$, $f + g$, $f \cdot g$, $\frac{f}{g}$ are defined in the usual way. For instance, $f \cdot g : A \to \mathbb{R}$ is defined as $(f \cdot g)(x) = f(x) \cdot g(x)$ for all $x \in A$.

Example 8.1. For an arbitrary real number c we define the function $f(x) = c$, $x \in \mathbb{R}$. Then $f \in \mathrm{C}(\mathbb{R})$.

Example 8.2. Let $f(x) = x$, $x \in \mathbb{R}$. Then $f \in \mathrm{C}(\mathbb{R})$. Indeed, to show this, let us use e.g. Definition 8.3. We fix any $a \in \mathbb{R}$. Then we obtain that for every $\varepsilon > 0$ there exists $\delta := \varepsilon > 0$ such that for each $x \in B(a, \delta)$ $|f(x) - f(a)| = |x - a| < \delta = \varepsilon$. So, f is continuous at a. Since a was an arbitrary point of $\mathbb{R}$, f is continuous on $\mathbb{R}$.

Example 8.3. Let $P(x) = a_0 x^m + a_1 x^{m-1} + \ldots + a_{m-1} x + a_m$, $x \in \mathbb{R}$, where $m \in \mathbb{N} \cup \{0\}$ and $a_0, a_1, \ldots, a_m$ are some real numbers. The function P is called a **polynomial function**. Theorem 8.1 and examples 8.1, 8.2 imply that $P \in \mathrm{C}(\mathbb{R})$.

Example 8.4. Let P and Q be two polynomial functions. We define the function $R(x) = \frac{P(x)}{Q(x)}$, $x \in \{z \in \mathbb{R} : Q(z) \neq 0\}$, which is called a **rational function**. By Theorem 8.1 and Example 8.3, the rational function R is continuous at any point where it is well-defined.

Example 8.5. The functions sin and cos are continuous on $\mathbb{R}$. The functions tan and cot are continuous on the set where they are well-defined. The continuity of functions sin and cos follows from Example 6.6. For the functions tan and cot the continuity follows from Theorem 8.1 and the equalities $\tan x = \frac{\sin x}{\cos x}$ and $\cot x = \frac{\cos x}{\sin x}$.

Example 8.6. Let $a > 0$ and $f(x) = a^x$, $x \in \mathbb{R}$. Then $f \in \mathrm{C}(\mathbb{R})$.

Exercise 8.1. Prove that the function from Example 8.6 is continuous on $\mathbb{R}$.

Exercise 8.2. Compute the following limits:
a) $\lim\limits_{x \to 0} (\tan x - e^x)$; b) $\lim\limits_{x \to 2} \frac{x^2 - 3^x + 1}{x - \sin \pi x}$; c) $\lim\limits_{x \to 3} \frac{x \cos x + 1}{x^3 + 1}$.

Exercise 8.3. Let a, b be a real numbers, $f(x) = x + 1$, $x \leq 0$ and $f(x) = ax + b$, $x > 0$. a) For which a, b the function f is monotone on $\mathbb{R}$? b) For which a, b the function f is continuous on $\mathbb{R}$?

Exercise 8.4. Let $f(x) = \lfloor x \rfloor \sin \pi x$, $x \in \mathbb{R}$. Prove that $f \in \mathrm{C}(\mathbb{R})$ and sketch its graph.

(*Hint:* If $x \in [k, k+1)$ for some $k \in \mathbb{Z}$, then $\lfloor x \rfloor = k$ and $f(x) = k \sin \pi x$. Find $f(k-)$ and $f(k+)$ at the points k.)

Exercise 8.5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function on $\mathbb{R}$ and $f(r) = r^3 + r + 1$ for all $r \in \mathbb{Q}$. Find the function f.

Exercise 8.6. Show that $|f| \in \mathrm{C}(A)$, if $f \in \mathrm{C}(A)$, where $|f|(x) := |f(x)|$, $x \in A$.

Exercise 8.7. For functions $f, g \in \mathrm{C}(A)$ we set $h(x) := \min\{f(x), g(x)\}$, $x \in A$, and $l(x) := \max\{f(x), g(x)\}$, $x \in A$. Prove that $h, l \in \mathrm{C}(A)$.

(*Hint:* Use the equalities $\min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$ and $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$.)

Definition 8.6. If a function $f : A \to \mathbb{R}$ is not continuous at a point $a \in A$, then f is said to be **discontinuous at the point** a.

Example 8.7. The function sgn, defined in Example 7.5, is continuous on $\mathbb{R} \setminus \{0\}$ and discontinuous at 0.

Exercise 8.8. Prove that the function $f(x) = \sin \frac{1}{x}$, $x \neq 0$, and $f(0) = 0$, is discontinuous at 0.

Exercise 8.9. Show that the **Dirichlet function** $f(x) = 1$, $x \in \mathbb{Q}$, and $f(x) = 0$, $x \in \mathbb{R} \setminus \mathbb{Q}$ is discontinuous at any point of $\mathbb{R}$.

8.2 Some Properties of Continuous Functions

Theorem 8.2. *Let a function $f : A \to \mathbb{R}$ be continuous at $a \in A$ and $f(a) < q$. Then*

$$\exists \delta > 0 \ \forall x \in A \cap B(a, \delta) : \ f(x) < q.$$

Proof. Using Definition 8.3, we obtain that for $\varepsilon := q - f(a) > 0$ there exists $\delta > 0$ such that for all $x \in A \cap B(a, \delta)$ $|f(x) - f(a)| < \varepsilon = q - f(a)$. In particular, $f(x) - f(a) < q - f(a)$, which implies $f(x) < q$ for all $x \in A \cap B(a, \delta)$. □

Theorem 8.3 (Limit of composition)**.** *Let a be a limit point of A (which could be $+\infty$ or $-\infty$) and let for a function $f : A \to \mathbb{R}$ there exists a limit $\lim_{x\to a} f(x) = p \in \mathbb{R}$. We also assume that $f(A) \cap \{p\} \subset B$ and a function $g : B \to \mathbb{R}$ is continuous at the point p. Then $\lim_{x\to a} g(f(x)) = g(p)$, that is, $\lim_{x\to a} g(f(x)) = g\left(\lim_{x\to a} f(x)\right)$.*

Proof. For any sequence $(x_n)_{n\geq 1}$ satisfying properties 1) and 2) from the definition of limit (see Definition 6.6), one has $f(x_n) \to p$, $n \to \infty$. Since g is continuous, $g(f(x_n)) \to g(p)$, $n \to \infty$, by Definition 8.2. □

Theorem 8.4 (Continuity of composition)**.** *We assume that $f : A \to \mathbb{R}$ is continuous at $a \in A$, $f(A) \subset B$ and a function $g : B \to \mathbb{R}$ is continuous at the point $f(a)$. Then the function $g \circ f$ is continuous at the point a.*

Proof. The statement immediately follows from Theorem 8.3, setting $p := f(a)$. □

Let $(a, b) \subset \mathbb{R}$, where $-\infty \leq a < b \leq +\infty$. Let $f : (a, b) \to \mathbb{R}$ be an increasing function. By Theorem 7.9 (ii), there exists $\lim_{x\to a+} f(x) =: c \in \mathbb{R}$, if f is bounded below. If f is unbounded below, then it is easy to see that $\lim_{x\to a+} f(x) = -\infty =: c$. Consequently, $\lim_{x\to a+} f(x) = c$ can be well defined for any increasing function. Similarly, $\lim_{x\to b-} f(x) =: d \leq +\infty$ is also well defined.

Theorem 8.5 (Existence of continuous inverse function)**.** *Let a function $f : (a, b) \to \mathbb{R}$ satisfy the following properties:*

1) f strictly increases on (a, b), that is, for any $x_1, x_2 \in (a, b)$ $x_1 < x_2$ implies $f(x_1) < f(x_2)$;

2) $f \in \mathrm{C}((a, b))$.

We set $c := \lim_{x\to a+} f(x)$ and $d := \lim_{x\to b-} f(x)$.
Then there exists a function $g : (c, d) \to (a, b)$ such that

a) g is strictly increasing on (c, d);

b) $g \in \mathrm{C}((c, d))$;

c) $g(f(x)) = x$ for all $x \in (a, b)$, and $f(g(y)) = y$ for all $y \in (c, d)$, that is, $g = f^{-1}$.

Remark 8.3. A similar statement also is true for a strictly decreasing function $f : (a, b) \to \mathbb{R}$, i.e. for a function such that for any $x_1, x_2 \in (a, b)$ $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

Remark 8.4. If $a \in \mathbb{R}$, then Theorem 8.5 is also valid for the set $[a, b)$.

8.3 Some Inverse Functions

Example 8.8. n-th root function $g(y) = \sqrt[m]{y}$, $y \geq 0$.
Let $m \in \mathbb{N}$ be fixed. We set $[a, b) = [0, +\infty)$ and $f(x) = x^m$, $x \in [0, +\infty)$. The function f satisfies conditions of Theorem 8.5, namely, it strictly increases and is continuous on $[0, +\infty)$. Moreover, $c = \lim\limits_{x\to 0+} x^m = 0$ and $d = \lim\limits_{x\to+\infty} x^m = +\infty$. Thus, according to Theorem 8.5, there exists a function $g : [0, +\infty) \to [0, +\infty)$ which increases and is continuous on $[0, +\infty)$ and inverse to f. Usually, the function g is denoted as follows $\sqrt[m]{y} = y^{\frac{1}{m}} := g(y)$, $y \geq 0$. Moreover, $\sqrt[m]{x^m} = x$ for each $x \geq 0$ and $(\sqrt[m]{y})^m = y$ for each $y \geq 0$.

Example 8.9. Logarithmic function $g(y) = \log_p y$, $y > 0$.
Let $p > 0$, $p \neq 1$ and $f(x) = p^x$, $x \in \mathbb{R}$. We want to prove that the function f has the inverse function, which is called the **logarithm**. We will consider the case $p > 1$, for which the function f is strictly increasing and continuous, by Example 8.6. Moreover, $c = \lim\limits_{x\to-\infty} p^x = 0$ and $d = \lim\limits_{x\to+\infty} p^x$. By Theorem 8.5, there exists a function $g : (0, +\infty) \to \mathbb{R}$, which is continuous on $(0, +\infty)$ and inverse to f. The function g is denoted by $\log_p y := g(y)$, $y > 0$, and it satisfies $\log_p p^x = x$ for all $x \in \mathbb{R}$ and $p^{\log_p y} = y$ for all $y > 0$.

Example 8.10. Trigonometric functions arcsin, arccos, arctan, arccot.
Let $[a, b] = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $f(x) = \sin x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. By the definition of sin, it is strictly increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Furthermore, by Example 8.5, sin is continuous on $\mathbb{R}$ and, in particular, on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus, using Theorem 8.5, there exists the continuous inverse function $g : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to f. It is denoted by $\arcsin y := g(y)$, $y \in [-1, 1]$, and satisfies $\arcsin(\sin x) = x$ for all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin(\arcsin y) = y$ for all $y \in [-1, 1]$.

Similarly, one can define the functions $\arccos : [-1, 1] \to [0, \pi]$, $\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\operatorname{arccot} : \mathbb{R} \to (0, \pi)$, which are inverse to $\cos : [0, \pi] \to [-1, 1]$, $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ and $\cot : (0, \pi) \to \mathbb{R}$, respectively. Moreover, each function is continuous on the set where it is defined.

Exercise 8.10. Sketch the graphs of the functions $\ln = \log_e$, $\log_{\frac{1}{2}}$, arcsin, arccos, arctan and arccot.

Exercise 8.11. Compute the following limits:
a) $\lim\limits_{x\to0} \frac{\ln(1+x)+\arcsin x^2}{\arccos x+\cos x}$; b) $\lim\limits_{x\to1} \frac{\arctan x}{1+\arctan x^2}$; c) $\lim\limits_{x\to0} \frac{\arcsin x}{x}$; d) $\lim\limits_{x\to0} \frac{x}{\sin x+\arcsin x}$; e) $\lim\limits_{x\to0} \frac{\arctan x}{x}$;
f) $\lim\limits_{x\to0} \frac{\arccos x-\frac{\pi}{2}}{x}$; g) $\lim\limits_{x\to0} \frac{\sin(\arctan x)}{\tan x}$.

8.4 Some Important Limits

Theorem 8.6. *Let $a > 0$ and $a \neq 1$. Then*

$$\lim_{x\to0} \frac{\log_a(1 + x)}{x} = \log_a e,$$

in particular, for $a = e$

$$\lim_{x\to0} \frac{\ln(1 + x)}{x} = 1.$$

Proof. We are going to use Theorem 8.3 about limit of composition in order to prove the needed equality. Let $A = (-1, +\infty)$, $f(x) = (1 + x)^{\frac{1}{x}}$, $x > -1$, $p = \lim\limits_{x\to0}(1 + x)^{\frac{1}{x}} = e > 0$; $B = (0, +\infty)$ and

$g(y) = \log_a y$, $y > 0$. In Example 8.9, we have proved that g is continuous and, consequently, it is continuous at $p = e > 0$. Thus, using Theorem 8.3, we obtain

$$\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \lim_{x\to 0} \log_a(1+x)^{\frac{1}{x}} = \log_a \left(\lim_{x\to 0}(1+x)^{\frac{1}{x}} \right) = \log_a e.$$

□

Theorem 8.7. *Let $a > 0$. Then*

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a,$$

in particular, for $a = e$

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

Proof. If $a = 1$, then the statement is true. We assume that $a \neq 1$. By the continuity and monotonicity of the function $h(x) = a^x$ (see Example 8.6), one can easily seen that $z := a^x - 1 \to 0$ provided $x \to 0$. Moreover, $x = \log_a(1 + z)$. Hence, by Theorem 8.6, we obtain

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{z\to 0} \frac{z}{\log_a(1+z)} = \frac{1}{\log_a e} = \log_e a = \ln a.$$

□

Theorem 8.8. *Let $\alpha \in \mathbb{R}$. Then*

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha.$$

Proof. For $\alpha = 0$ the statement holds. We assume that $\alpha \neq 0$. Using the continuity of ln (see Example 8.9), we have $\ln(1 + x) \to \ln 1 = 0$, $x \to 0$. By theorems 8.1, 8.6 and 8.7, we get

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \lim_{x\to 0} \frac{\left(e^{\alpha \ln(1+x)} - 1\right) \alpha \ln(1+x)}{x\alpha \ln(1+x)} = \alpha \lim_{x\to 0} \frac{e^{\alpha \ln(1+x)} - 1}{\alpha \ln(1+x)} \cdot \lim_{x\to 0} \frac{\ln(1+x)}{x} = \alpha.$$

□

Theorem 8.9. *Let $\alpha \in \mathbb{R}$ and $f(x) = x^\alpha$, $x > 0$. Then f is continuous on $(0, +\infty)$.*

Proof. Since for each $x > 0$, one has $f(x) = e^{\alpha \ln x}$, the statement follows from the continuities of the exponential function and the logarithm (see examples 8.6 and 8.9, respectively) and Theorem 8.4. □

Exercise 8.12. Compute the following limits:
a) $\lim_{x\to 0}(\cos x)^x$; b) $\lim_{x\to +\infty} x(\ln(1+x) - \ln x)$; c) $\lim_{x\to 0}\left(\frac{1+\sin 2x}{\cos 2x}\right)^{\frac{1}{x}}$; d) $\lim_{x\to 0} \frac{1-\cos x}{1-\cos 2x}$; e) $\lim_{x\to 0} \frac{\ln(1+x)+e^x-\cos x}{e^{x^2}-1+\sin x}$; f) $\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}$; g) $\lim_{x\to 0} \frac{\arcsin(x-1)}{x^m - 1}$ for $m \in \mathbb{N}$; h) $\lim_{x\to 0} \frac{1-(\cos mx)^m}{x^2}$ for $m \in \mathbb{N}$; i) $\lim_{x\to 0} \frac{1-(\cos mx)^{\frac{1}{m}}}{x^2}$ for $m \in \mathbb{N}$; k) $\lim_{x\to 1} \frac{(\sin(\pi \cdot 2^x))^2}{\ln(\cos(\pi \cdot 2^x))}$; l) $\lim_{x\to 0}\left(\frac{1+x\cdot 2^x}{1+x\cdot 3^x}\right)^{\frac{1}{x^2}}$; m) $\lim_{x\to 0} \frac{x^x - a^a}{x-a}$, for $a > 0$; n) $\lim_{x\to 1}(1-x)\log_x 2$.

References

[1] K.A. Ross. *Elementary Analysis: The Theory of Calculus*. Undergraduate Texts in Mathematics. Springer New York, 2013.