4 Lecture 4 – Subsequences and Monotone Sequences

4.1 Monotone Sequences

The main goal of this section is to prove that any bounded monotone sequence must converge. So, we start from the definition.

Definition 4.1. A sequence $(a_n)_{n\geq 1}$ of real numbers is called an **increasing sequence** if $a_n \leq a_{n+1}$ for all $n \geq 1$, and $(a_n)_{n\geq 1}$ is called a **decreasing sequence** if $a_n \geq a_{n+1}$ for all $n \geq 1$. A sequence that is increasing or decreasing is said to be a **monotone sequence**.

Example 4.1. The sequence $(1, 1, 2, 2, 3, 3, 4, 4, \ldots)$ is increasing, but $(-1, 1, -1, 1, \ldots)$ is not monotone.

Exercise 4.1. a) Show that any bounded above increasing sequence is bounded. b) Show that any bounded below decreasing sequence is bounded.

Exercise 4.2. a) Prove that $(n2^{-n})_{n\geq 2}$ is a decreasing sequence.

b) Let $(a_n)_{n\geq 1}$ be an increasing sequence of positive numbers and define $\sigma_n = \frac{a_1+\ldots+a_n}{n}$. Prove that $(\sigma_n)_{n\geq 1}$ is also an increasing sequence.

Theorem 4.1. *Every bounded monotone sequence converges.*

Proof. We will prove the theorem for increasing sequences. The case of decreasing sequences is left to Exercise 4.3. So, let a sequence $(a_n)_{n\geq 1}$ increase. By the assumption of the theorem, $(a_n)_{n\geq 1}$ is bounded, that is, there exists $C \in \mathbb{R}$ such that $|a_n| \leq C$ for all $n \geq 1$. This implies that the set $A := \{a_n : n \geq 1\}$ is also bounded. Thus, by Theorem 2.2 (i) there exists $\sup A =: \sup\limits_{n\geq 1} a_n$ denoted by a. Let us prove that $a_n \to a$, $n \to \infty$. We first note that $a_n \leq a$ for all $n \geq 1$, since the supremum of A is also its upper bound (see Definition 2.6). Next, we take an arbitrary $\varepsilon > 0$ and use Theorem 2.1 (i). So, there exists a number m such that $a_m > a - \varepsilon$. By the monotonicity, $a - \varepsilon < a_m \leq a_n$ for all $n \geq m$. Thus, setting $N := m$, one has $a - \varepsilon < a_n \leq a$ for all $n \geq N$ which implies $|a_n - a| < \varepsilon$. □

Exercise 4.3. Prove Theorem 4.1 for decreasing sequences.

Remark 4.1. Theorem 4.1 remains valid if one requires the monotonicity of $(a_n)_{n\geq 1}$ starting from some number m, that is, the monotonicity of $(a_n)_{n\geq m} = (a_m, a_{m+1}, \ldots)$.

Example 4.2. Prove that $\lim\limits_{n\to\infty} \frac{10^n}{n!} = 0$, where $n! := 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.

Solution. First we note that $\frac{10^{n+1}}{(n+1)!} < \frac{10^n}{n!} \Leftrightarrow 10 < n+1 \Leftrightarrow n > 9$. Hence, the sequence $\left(\frac{10^n}{n!}\right)_{n\geq 10}$ is decreasing. Moreover, it is bounded below by zero. Thus, $\left(\frac{10^n}{n!}\right)_{n\geq 10}$ is bounded, by Exercise 4.1 b). Using Theorem 4.1, one gets that there exists $a \in \mathbb{R}$ such that $\lim\limits_{n\to\infty} \frac{10^n}{n!} = a$. But we can write $\frac{10^{n+1}}{(n+1)!} = \frac{10^n}{n!} \cdot \frac{10}{n+1}$. So,

$$a = \lim_{n\to\infty} \frac{10^{n+1}}{(n+1)!} = \lim_{n\to\infty} \frac{10^n}{n!} \cdot \lim_{n\to\infty} \frac{10}{n+1} = a \cdot 0.$$

This implies $a = 0$.

Exercise 4.4. Show that a) $\lim_{n\to\infty} \frac{n!}{2^{n^2}} = 0$; b) $\lim_{n\to\infty} \frac{n}{2^{\sqrt{n}}} = 0$.

Exercise 4.5. Find a limit of the sequence $\left(\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \dots\right)$.

Exercise 4.6. Let $a_1 = 1$ and $a_{n+1} = \frac{1}{3}(a_n + 1)$ for all $n \geq 1$.

a) Find a_2, a_3, a_4.

b) Use induction to show that $a_n > \frac{1}{2}$ for all $n \geq 1$.

c) Show that $(a_n)_{n\geq 1}$ is a decreasing sequence.

d) Show that $\lim_{n\to\infty} a_n$ exists and find it.

Exercise 4.7. Let $c > 0$, $a_1 > 0$ and let $a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right)$ for all $n \geq 1$.

a) Show that $a_n \geq \sqrt{c}$ for all $n \geq 2$.

b) Show that $(a_n)_{n\geq 2}$ is a decreasing sequence.

c) Show that $\lim_{n\to\infty} a_n$ exists and find it.

Theorem 4.2. *(i) If $(a_n)_{n\geq 1}$ is an unbounded increasing sequence, then $\lim_{n\to\infty} a_n = +\infty$.*

(ii) If $(a_n)_{n\geq 1}$ is an unbounded decreasing sequence, then $\lim_{n\to\infty} a_n = -\infty$.

Proof. We will prove only Part (i) of the theorem. The proof of Part (ii) is similar. If $(a_n)_{n\geq 1}$ is an unbounded increasing sequence, then it must be unbounded above, since it is bounded below by a_1. Taking any C and using the unboundedness of $(a_n)_{n\geq 1}$, one can find a number $m \in \mathbb{N}$ such that $a_m \geq C$. Next, by the monotonicity of $(a_n)_{n\geq 1}$, the inequality $a_n \geq a_m \geq C$ trivially holds for all $n \geq N := m$. This proves $\lim_{n\to\infty} a_n = +\infty$ (see Definition 3.4). □

Corollary 4.1. *If $(a_n)_{n\geq 1}$ is a monotone sequence, then the sequence either converges, diverges to $+\infty$, or diverges to $-\infty$. Thus $\lim_{n\to\infty} a_n$ is always meaningful for monotone sequences.*

Proof. The proof immediately follows from theorems 4.1 and 4.2. □

Exercise 4.8. Let A be a bounded nonempty subset of $\mathbb{R}$ such that $\sup A$ is not in A. Prove that there is an increasing sequence $(a_n)_{n\geq 1}$ of points from A such that $\lim_{n\to\infty} a_n = \sup A$.

4.2 The number e

In this section, we will consider two sequences of positive numbers

$$\left(a_n := \left(1 + \frac{1}{n}\right)^n\right)_{n\geq 1} \quad \text{and} \quad \left(b_n := \left(1 + \frac{1}{n}\right)^{n+1}\right)_{n\geq 1} \tag{1}$$

and study their properties.

Theorem 4.3. *The sequences defined in (1) satisfy the following properties:*

1) $a_n < b_n$ for all $n \geq 1$;

2) the sequence $(a_n)_{n\geq 1}$ increases;

3) the sequence $(b_n)_{n\geq 1}$ decreases.

Proof. Since $b_n = a_n \left(1 + \frac{1}{n}\right) = a_n + \frac{a_n}{n} > a_n$ for all $n \geq 1$, Property 1) is proved.

To prove 2), we are going to use Bernoulli's inequality (see Theorem 2.6). So, one has

$$\frac{a_n}{a_{n-1}} = \left(\frac{n+1}{n}\right)^n \left(\frac{n-1}{n}\right)^{n-1} = \frac{n}{n-1}\left(1 - \frac{1}{n^2}\right)^n > \frac{n}{n-1}\left(1 - \frac{n}{n^2}\right) = 1,$$

for all $n \geq 2$. Thus, $a_n > a_{n-1}$ for all $n \geq 2$.

For the prove of 3) we use the same argument. We consider

$$\begin{aligned}\frac{b_{n-1}}{b_n} &= \left(\frac{n}{n-1}\right)^n \left(\frac{n}{n+1}\right)^{n+1} = \frac{n-1}{n}\left(\frac{n^2}{n^2-1}\right)^{n+1} \\ &= \frac{n-1}{n}\left(1 + \frac{1}{n^2-1}\right)^{n+1} > \frac{n-1}{n}\left(1 + \frac{n+1}{n^2-1}\right) = 1,\end{aligned}$$

for all $n \geq 2$. Hence, $b_{n-1} > b_n$ for all $n \geq 2$. □

Theorem 4.3 yields the following inequalities

$$a_1 < a_2 < \ldots < a_n < \ldots < b_n < \ldots < b_2 < b_1. \tag{2}$$

Consequently, the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are monotone and bounded. By Theorem 4.1, they converge. We set

$$e := \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2,718281828459045...$$

It is known that e is an irrational number. The number e is one of the most important constants in mathematics.

Since $b_n = a_n \left(1 + \frac{1}{n}\right)$ for all $n \geq 1$, one has $b_n \to e$, $n \to \infty$. We also note that

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}, \tag{3}$$

by inequalities (2).

Definition 4.2. The logarithm to base e is called the **natural logarithm** and is denoted by $\ln := \log_e$, that is, for each $a > 0$ $\ln a$ is a (unique!) real number such that $e^{\ln a} = a$.

The inequality (3) immediately implies

$$\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$$

for all $n \geq 1$.

Exercise 4.9. Show that $\lim\limits_{n\to\infty}\left(n \ln\left(1 + \frac{1}{n}\right)\right) = 1$.

Exercise 4.10. Prove that for each $x > 0$ the sequence $\left(\left(1 + \frac{x}{n}\right)^n\right)_{n\geq 1}$ is increasing and bounded.

4.3 Subsequences

4.3.1 Subsequences and Subsequential Limits

Let $(a_n)_{n\geq 1}$ be a sequence. We consider any subsequence $(n_k)_{k\geq 1}$ of natural numbers such that $1 \leq n_1 < n_2 < \ldots < n_k < n_{k+1} < \ldots$. We note that $n_k \geq k$ and $n_k \to +\infty$, $k \to \infty$.

Example 4.3. 1) $n_k = k$, $k \geq 1$; then $(n_k)_{k\geq 1} = (1, 2, 3, \ldots, k, \ldots)$;

2) $n_k = 2k$, $k \geq 1$; then $(n_k)_{k\geq 1} = (2, 4, 6, \ldots, 2k, \ldots)$;

3) $n_k = k^2$, $k \geq 1$; then $(n_k)_{k\geq 1} = (1, 2, 9, \ldots, k^2, \ldots)$;

4) $n_k = 2^k$, $k \geq 1$; then $(n_k)_{k\geq 1} = (2, 4, 8, \ldots, 2^k, \ldots)$.

Definition 4.3. A sequence $(a_{n_k})_{k\geq 1} = (a_{n_1}, a_{n_2}, a_{n_3}, \ldots, a_{n_k}, \ldots)$ is said to be a **subsequence** of $(a_n)_{n\geq 1}$.

Thus, $(a_{n_k})_{k\geq 1}$ is just a selection of some (possibly all) of the a_n's taken in order.

Remark 4.2. The following properties follows from the definition of subsequence.

1. If a sequence is bounded, then every its subsequence is bounded.
2. If a sequence converges to a (that could be $+\infty$ or $-\infty$), then every its subsequence also converges to a.

Exercise 4.11. Prove that a monotone sequences which contains a bounded subsequence is bounded.

Exercise 4.12. Prove that a sequence $(a_n)_{n\geq 1}$ converges iff $(a_{2k})_{k\geq 1}$, $(a_{2k-1})_{k\geq 1}$ and $(a_{3k})_{k\geq 1}$ converge.

Definition 4.4. A **subsequential limit** of a sequence $(a_n)_{n\geq 1}$ is any real number or the symbol $+\infty$ or $-\infty$ that is the limit of some subsequence of $(a_n)_{n\geq 1}$. Let A denotes the **set of all subsequential limit** of $(a_n)_{n\geq 1}$.

Example 4.4. a) For the sequence $(1, 2, 3, \ldots, n, \ldots)$ the set of all subsequential limit $A = \{+\infty\}$.

b) For the sequence $(-1, 1, -1, \ldots, (-1)^n, \ldots)$ the set of all subsequential limit $A = \{-1, 1\}$.

c) If $a_n \to a$, then $A = \{a\}$, by Remark 4.2.

Exercise 4.13. Prove the following statements.
a) $-\infty \in A \quad \Leftrightarrow \quad (a_n)_{n\geq 1}$ is unbounded below. b) $+\infty \in A \quad \Leftrightarrow \quad (a_n)_{n\geq 1}$ is unbounded above.

Exercise 4.14. Find the set A of all subsequential limits of the following sequences.
a) $(\sin 3\pi n)_{n\geq 1}$; b) $(\sin \alpha\pi n)_{n\geq 1}$ for $\alpha \in \mathbb{Q}$; c) $(a_n)_{n\geq 1}$, where $a_n = \begin{cases} (-1)^{\frac{n+1}{2}} + n, & \text{if } n \text{ is odd,} \\ (-1)^{\frac{n}{2}} + \frac{1}{n}, & \text{if } n \text{ is even.} \end{cases}$

4.3.2 Existence of Monotone Subsequence

Theorem 4.4. *A number $a \in \mathbb{R}$ is a subsequential limit of a sequence $(a_n)_{n\geq 1}$ iff*

$$\forall \varepsilon > 0 \ \forall N \in \mathbb{N} \ \exists \tilde{n} \in \mathbb{N}: \ \tilde{n} \geq N, \ |a_{\tilde{n}} - a| < \varepsilon. \tag{4}$$

Proof. We first prove the necessity. Let $a \in A$. Then there exists a subsequence $(a_{n_k})_{k\geq 1}$ such that $a_{n_k} \to a$, $k \to \infty$. We fix an arbitrary $\varepsilon > 0$ and $N \in \mathbb{N}$. By the definition of the limit, $\exists K_1 \in \mathbb{N} \ \forall k \geq K_1 : |a_{n_k} - a| < \varepsilon$. Similarly, $\exists K_2 \in \mathbb{N} \ \forall k \geq K_2 : n_k \geq N$. Thus, taking $\tilde{k} := \max\{K_1, K_2\}$, $\tilde{n} := n_{\tilde{k}}$, one has $\tilde{n} \geq N$ and $|a_{\tilde{n}} - a| < \varepsilon$.

To prove the sufficiency, we are going to construct a subsequence of $(a_n)_{n\geq 1}$ converging to a. Let (4) holds. Then, by (4), for $\varepsilon = 1$ and $N = 1$ there exists $n_1 \geq 1$ such that $|a_{n_1} - a| < 1$. Similarly, for $\varepsilon = \frac{1}{2}$ and $N = n_1 + 1$ there exists $n_2 \geq n_1 + 1$ such that $|a_{n_2} - a| < \frac{1}{2}$ and so on. Consequently, we obtain a subsequence $(a_{n_k})_{k\geq 1}$ satisfying $|a_{n_k} - a| < \frac{1}{k}$ for all $k \geq 1$. Using Theorem 3.7 and Exercise 3.5 a), one can see that $a_{n_k} \to a$, $k \to \infty$. □

Exercise 4.15. Show that $+\infty \in A$ ($-\infty \in A$) provided $\forall C \in \mathbb{R} \ \forall N \in \mathbb{N} \ \exists \tilde{n} \in \mathbb{N}: \ \tilde{n} \geq N$ and $a_{\tilde{n}} \geq C$ ($a_{\tilde{n}} \leq C$).

Theorem 4.5. *Every sequence of real numbers contains a monotone subsequence.*

Proof. We consider the set $M := \{n \in \mathbb{N} : \ \forall m > n \quad a_m > a_n\}$. If M is infinite, then M can be written as $M = \{n_1, n_2, \ldots, n_k, \ldots\}$, where $n_1 < n_2 < \ldots < n_k < \ldots$. By the definition of M, we have $a_{n_1} < a_{n_2} < \ldots < a_{n_k} < \ldots$. So, the subsequence $(a_{n_k})_{k\geq 1}$ increases.

If M is finite, then let n_1 be the smallest natural number such that $\forall m \geq n_1 : \ m \notin M$. Since $n_1 \notin M$, one can find $n_2 > n_1$ such that $a_{n_1} \geq a_{n_2}$. Similarly, since $n_2 \notin M$, one can find $n_3 > n_2$ such that $a_{n_2} \geq a_{n_3}$ and so on. Thus, the constructed subsequence $(a_{n_k})_{k\geq 1}$ decreases. □

Corollary 4.2. *For every sequence the set of its subsequential limits is not empty.*

Proof. The corollary immediately follows from Theorem 4.5 and Corollary 4.1. □

Theorem 4.6 (Bolzano-Weierstrass theorem). *Every bounded sequence has a convergent subsequence.*

Proof. The theorem is a direct consequence of theorems 4.5 and 4.1. □

References

[1] K.A. Ross. *Elementary Analysis: The Theory of Calculus.* Undergraduate Texts in Mathematics. Springer New York, 2013.