24 Lecture 24 – Basis

24.1 Linear Independence

In this section, we are going to define the notion of linear independence of a list of vectors.

Definition 24.1. Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are called **linearly independent** if the only solution for $a_1, a_2, \ldots, a_n \in \mathbb{F}$ to the equation

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n = 0$$

is $a_1 = a_2 = \ldots = a_n = 0$. Otherwise, the vectors $v_1, v_2, \ldots, v_n$ are said to be **linearly dependent**.

Example 24.1. The vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ from Example 23.4 are linearly independent, since

$$a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \ldots + a_n\mathbf{e}_n = (a_1, a_2, \ldots, a_n) = (0, 0, \ldots, 0)$$

provided $a_1 = a_2 = \ldots = a_n = 0$.

Example 24.2. The vectors $\mathbf{v}_1 = (1, 1, 3)$, $\mathbf{v}_2 = (1, 1, 0)$, $\mathbf{v}_3 = (0, 0, 1)$ are linearly dependent because

$$\mathbf{v}_1 - \mathbf{v}_2 - 3\mathbf{v}_3 = (0, 0, 0).$$

Example 24.3. The vectors $(1, z, z^2, \ldots, z^n)$ in $\mathbb{F}_n[z]$ are linearly independent.

Exercise 24.1. Show that the vectors $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 2, 3)$, and $\mathbf{v}_3 = (2, -1, 1)$ are linearly independent in $\mathbb{R}^3$. Write $\mathbf{v} = (1, -2, 5)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

Exercise 24.2. Consider the complex vector space $V = \mathbb{C}^3$ and the vectors $\mathbf{v}_1 = (i, 0, 0)$, $\mathbf{v}_2 = (i, 1, 0)$, $\mathbf{v}_3 = (i, i, -1)$.
a) Prove that $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = V$.
b) Are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ a basis of $\mathbb{C}^3$?

Exercise 24.3. Determine the value of $\lambda \in \mathbb{R}$ for which each vectors $(\lambda, -1, -1)$, $(-1, \lambda, -1)$, $(-1, -1, \lambda)$ are linearly dependent in $\mathbb{R}^3$.

Theorem 24.1. *Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent iff each vector $\mathbf{v} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ can be unequally written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.*

Proof. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be linearly independent. If $\mathbf{v} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ can be written as

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n = a_1'\mathbf{v}_1 + a_2'\mathbf{v}_2 + \ldots + a_n'\mathbf{v}_n,$$

then $\mathbf{0} = \mathbf{v} - \mathbf{v} = (a_1 - a_1')\mathbf{v}_1 + (a_2 - a_2')\mathbf{v}_2 + \ldots + (a_n - a_n')\mathbf{v}_n$, which implies that $a_1 = a_1'$, $a_2 = a_2'$, ..., $a_n = a_n'$. The sufficiency can be proved trivially, taking $\mathbf{v} = \mathbf{0}$. □

Theorem 24.2. *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$. Then there exists $j \in \{2, \ldots, n\}$ such that*

1) $\mathbf{v}_j \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}\}$;

2) $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

Proof. Since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent, there exist $a_1, a_2, \ldots, a_n \in \mathbb{F}$ such that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n = 0$. Since $\mathbf{v}_1 \neq \mathbf{0}$, not all of $a_2, \ldots, a_n$ are 0. Let $j \in \{2, \ldots, n\}$ be the largest index such that $a_j \neq 0$. Then we have

$$\mathbf{v}_j = -\frac{a_1}{a_j}\mathbf{v}_1 - \frac{a_2}{a_j}\mathbf{v}_2 - \ldots - \frac{a_{j-1}}{a_j}\mathbf{v}_{j-1}. \tag{48}$$

This implies 1).

Let $\mathbf{v}$ be an arbitrary vector from $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. It means that there exist $b_1, b_2, \ldots, b_n \in \mathbb{F}$ such that

$$v = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \ldots + b_n\mathbf{v}_n.$$

According to (48), $\mathbf{v}$ can be rewritten as linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n$. This proves 2). □

Theorem 24.3. *Let V be a finite-dimensional vector space, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be linearly independent and span V, and let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be vectors that span V. Then $n \leq m$.*

Proof. For the proof of the theorem see the proof of Theorem 5.2.9 [3]. □

Exercise 24.4. Let V be a vector space over $\mathbb{F}$, and suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ are linearly independent. Let $\mathbf{w}$ be a vector from V such that the vectors $\mathbf{v}_1 + \mathbf{w}, \mathbf{v}_2 + \mathbf{w}, \ldots, \mathbf{v}_n + \mathbf{w}$ are linearly dependent. Prove that $\mathbf{w} \in \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

24.2 Bases

Definition 24.2. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of a finite-dimensional vector space V if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent and span V, i.e. $V = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

Remark 24.1. We remark that each vector $\mathbf{v} \in V$ can be uniquely written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ iff $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V.

Example 24.4. The set of the vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis of $\mathbb{F}^n$.

Exercise 24.5. Prove that the set of vectors $(1, 1, 0)$, $(1, 0, 0)$, $(0, 0, 1)$ is a basis of $\mathbb{F}^3$.

Example 24.5. The set $1, z^2, \ldots, z^n$ is a basis of $\mathbb{F}_n[z]$.

Theorem 24.4 (Basis reduction theorem)**.** *If $V = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then either the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V or some $\mathbf{v}_k$ can be removed to obtain a basis of V.*

Proof. Suppose $V = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. We start with the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and sequentially run through all vectors $\mathbf{v}_k$ for $k = 1, 2, \ldots, m$ to determine whether to keep or remove them from S:

Step 1. If $\mathbf{v}_1 = \mathbf{0}$, then remove $\mathbf{v}_1$ from S. Otherwise, leave S unchanged.

Step k. If $\mathbf{v}_k \in \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}\}$, then remove $\mathbf{v}_k$ from S. Otherwise, leave S unchanged.

The final set S still spans V since, at each step, a vector was only discarded if it was already in the span of the previous vectors. The process also ensures that no vector is in the span of the previous vectors. Hence, by Theorem 24.2, the final list S is linearly independent. It follows that S is a basis of V. □

Example 24.6. The set of vectors $\mathbf{v}_1 = (1, -1, 0)$, $\mathbf{v}_2 = (0, 1, 0)$, $\mathbf{v}_3 = (1, 1, 1)$, $\mathbf{v}_4 = (0, -1, 2)$ are linearly dependent, since

$$0\mathbf{v}_1 + 3\mathbf{v}_2 - 2\mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}.$$

But the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis of $\mathbb{R}^3$. Indeed, each element $\mathbf{v} = (x, y, z)$ can be uniquely written as follows

$$(x, y, z) = (x - z)\mathbf{v}_1 + (x + y - 2z)\mathbf{v}_2 + z\mathbf{v}_3.$$

Corollary 24.1. *Every finite-dimensional vector space has a basis.*

Proof. The statement immediately follows from Theorem 24.4. □

Theorem 24.5 (Basis Extension Theorem)**.** *Every linearly independent set of vectors in a finite-dimensional vector space V can be extended to a basis of V.*

Proof. Let V be finite-dimensional and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be linearly independent. Since V is finite-dimensional, there exists a set of vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ that spans V. We are going to adjoin some of the $\mathbf{w}_k$ to $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in order to create a basis of V.

Step 1. If $\mathbf{w}_1 \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then let $S := \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Otherwise, $S := \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, w_1\}$.

Step k. If $\mathbf{w}_k \in \operatorname{span} S$, then leave S unchanged. Otherwise, adjoin $\mathbf{w}_k$ to S.

After each step, the set S is still linearly independent, since we only adjoined $\mathbf{w}_k$ if $\mathbf{w}_k$ was not in the span of the previous vectors. After m steps, $\mathbf{w}_k \in \operatorname{span} S$ for all $k = 1, 2, \ldots, m$. Since the set $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ spans V, S also spans V. Consequently, S is a basis of V. □

24.3 Dimension

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ be two bases of a finite-dimensional vector space V, that is, they both are linearly independent and span V. Then by Theorem 24.3, it follows that $n = m$.

Definition 24.3. We call the length of any basis of V the **dimension** of V and denote by $\dim V$.

Example 24.7. According to Example 24.4, the dimension of $\mathbb{F}^n$ equals n.

Example 24.8. By Example 24.5, the dimension of $\mathbb{F}_n[z]$ equals $n + 1$.

Exercise 24.6. Let $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbb{F}_n[z]$ satisfy $\mathbf{p}_j(2) = 0$ for all $j = 0, 1, \ldots, n$. Prove that $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_n$ must be a linearly dependent in $\mathbb{F}_n[z]$.

Remark 24.2. We note that $\dim \mathbb{C}^n = n$ as a complex vector space, whereas $\dim \mathbb{C}^n = 2n$ as a real vector space. This comes from the fact that we can view $\mathbb{C}$ itself as a real vector space of dimension 2 with basis $\{1, i\}$.

Theorem 24.6. *Let V be a finite-dimensional vector space with $\dim V = n$. Then*

(i) If $U \subset V$ is a subspace of V, then $\dim U \le \dim V$.

(ii) If $V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V.

(iii) If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent in V, then they form a basis of V.

Proof. To prove statement (i), first we note that U is necessarily finite-dimensional (otherwise we could find a list of linearly independent vectors longer than $\dim V$). Therefore, by Corollary 24.1, U has a basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ which are linearly independent in both U and V. By Theorem 24.5, we can extend $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ to a basis of V, which is of length n since $\dim V = n$. This implies that $m \le n$.

In order to prove statement (ii), we suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span V. Then, by the basis reduction theorem (see Theorem 24.4), this set can be reduced to a basis. However, every basis of V has length n. Hence, no vector needs to be removed from $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. It follows that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V.

To prove statement (iii), we assume that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent. By the basis extension theorem (see Theorem 24.5), this set can be extended to a basis. However, every basis has length n. Hence, no vector needs to be added to $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. It follows that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V. □

Theorem 24.7. *Let $U \subset V$ be a subspace of a finite-dimensional vector space V. Then there exists a subspace $W \subset V$ such that $V = U \oplus W$.*

Proof. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ be a basis of U. By Theorem 24.6 (i), we know that $m \le \dim V$. Hence, by the basis extension theorem (see Theorem 24.5), the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ can be extended to a basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ of V. Let $W := \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$.

We now show that $V = U \oplus W$. Since the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ is a basis of V, each element $\mathbf{v}$ of V can be uniquely written as follows

$$\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_m\mathbf{u}_m + b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + \ldots + b_k\mathbf{w}_k = \mathbf{u} + \mathbf{w}$$

for some $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_k \in \mathbb{F}$, where $\mathbf{u} := a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \ldots + a_m\mathbf{u}_m$ and $\mathbf{w} = b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + \ldots + b_k\mathbf{w}_k$. Since $\mathbf{u} \in U$ and $\mathbf{w} \in W$, V is the direct sum of U and W, according to Definition 23.3. □

Exercise 24.7. Let V be a finite-dimensional vector space over $\mathbb{F}$ with $\dim V = n$ for some $n \in \mathbb{N}$. Prove that there exist n one-dimensional subspaces $U_1, U_2, \ldots, U_n$ of V such that

$$V = U_1 \oplus U_2 \oplus \ldots \oplus U_n.$$

Theorem 24.8. *If $U, W \subset V$ are subspaces of a finite-dimensional vector space, then*

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

Proof. For the proof of the theorem see the proof of Theorem 5.4.6 [3]. □

Exercise 24.8. Let V be a finite-dimensional vector space over $\mathbb{F}$, and let U be a vector subspace of V for which $\dim U = \dim V$. Prove that $U = V$.