21 Lecture 21 – Complex Numbers

21.1 Definition and Basic Properties

We recall that the equation $az^2 + bz + c = 0$, $a, b, c \in \mathbb{R}$, $a \neq 0$, has solutions if and only if $D := b^2 - 4ac \geq 0$ which can be computed by the formula $z_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$. Thus, e.g. the equation

$$z^2 - 2z + 2 = 0 \tag{41}$$

has no solutions, since $D = 4 - 4 \cdot 1 \cdot 2 = -4 < 0$. However, we can formally take $z_1 := \frac{2+\sqrt{-4}}{2}$ and $z_2 := \frac{2-\sqrt{-4}}{2}$. If $\sqrt{-4}$ was a number such that $(\sqrt{-4})^2 = -4$, then a simple computation would show that z_1 and z_2 are solutions to (41). We are going to give this idea the rigorous meaning, namely, we extend the set of real numbers and later show that any polynomial equation has solutions in that class of numbers.

We consider a new symbol i and postulate that $i = \sqrt{-1}$, that is, $i^2 = -1$.

Definition 21.1. A number $z = x + yi$, where $x, y \in \mathbb{R}$ and $i^2 = -1$, is called a **complex number**. The number x is called the **real part** of z and is denoted by $x = \operatorname{Re} z$. The number y is called the **imaginary part** of z and is denoted by $y = \operatorname{Im} z$.

The set of all complex numbers is denoted by $\mathbb{C}$, i.e $\mathbb{C} = \{z = x + yi : \ x, y \in \mathbb{R}\}$.

Remark 21.1. If $\operatorname{Im} z = 0$, that is, $z = x + 0i$, then we will identify z with the real number x and write $z \in \mathbb{R}$.

Next, we introduce operations on complex numbers.

Addition and subtraction of complex numbers: For $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$ from $\mathbb{C}$ we define

$$z_1 \pm z_2 = (x_1 \pm x_2) + (y_1 \pm y_2)i. \tag{42}$$

Example 21.1. a) $(1-2i)+(2+4i) = (1+2)+(-2+4)i = 3+2i$; b) $i+(2-2i) = (0+2)+(1-2)i = 2-i$.

Exercise 21.1. Prove that $z_1 + z_2 = z_2 + z_1$ and $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ for all $z_1, z_2, z_3 \in \mathbb{C}$.

Multiplication and division of complex numbers: For $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$ from $\mathbb{C}$ we define

$$z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + (y_1 x_2 + x_1 y_2)i, \tag{43}$$

$$z_1 / z_2 = \frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2} i, \quad z_2 \neq 0. \tag{44}$$

Remark 21.2. The multiplication rule is motivated by the multiplication rule of polynomials and the equality $i^2 = -1$. Indeed, multiplying $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ as two polynomials, we have

$$z_1 \cdot z_2 = (x_1 + y_1 i) \cdot (x_2 + y_2 i) = x_1 x_2 + x_1 y_2 i + y_1 x_2 i + y_1 y_2 i^2 = (x_1 x_2 - y_1 y_2) + (y_1 x_2 + x_1 y_2)i.$$

Remark 21.3. The division of two complex numbers is motivated by the following observation: $(x_2 + y_2 i)(x_2 - y_2 i) = x_2^2 + y_2^2$. Thus, for $z_2 \neq 0$

$$\frac{z_1}{z_2} = \frac{x_1 + y_1 i}{x_2 + y_2 i} = \frac{(x_1 + y_1 i) \cdot (x_2 - y_2 i)}{(x_2 + y_2 i) \cdot (x_2 - y_2 i)} = \frac{(x_1 x_2 + y_1 y_2) + (y_1 x_2 - x_1 y_2)i}{x_2^2 + y_2^2}.$$

Example 21.2. a) $(2-i)(1+3i) = 2+6i-i-3i^2 = 2+5i-3\cdot(-1) = 5+5i$;
b) $\frac{1-i}{1-2i} = \frac{(1-i)(1+2i)}{(1-2i)(1+2i)} = \frac{1+2-i+2i}{1^2+2^2} = \frac{3+i}{5} = \frac{3}{5} + \frac{1}{5}i$; c) $\frac{1}{i} = \frac{1\cdot(-i)}{i\cdot(-i)} = -i$.

Exercise 21.2. Express the following complex numbers in the form $x+yi$ for $x, y \in \mathbb{R}$:
a) $(-2+3i)(1+i)$; b) $(\sqrt{2}-i)^2$; c) $\frac{3-i}{2+2i}$; d) $\frac{i}{(1-i)^2}$.

Exercise 21.3. Show that for $z_k = x_k + y_k i \in \mathbb{C}$, $k = 1, 2, 3$,
a) $z_1 \cdot z_2 = z_2 \cdot z_1$; b) $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$; c) $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$;
d) $z_1 \cdot z_2 \in \mathbb{R}$ if $z_1, z_2 \in \mathbb{R}$.

21.2 Complex Conjugate and Absolute Value of Complex Numbers

Definition 21.2. The number $\overline{z} := x - yi$ is sad to be the **conjugate** of a complex number $z = x + yi \in \mathbb{C}$.

Theorem 21.1. *Let $z_1, z_2, z \in \mathbb{C}$. Then the following equalities hold:*

a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$;

b) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$;

c) $z + \overline{z} = 2\operatorname{Re} z$ *and* $z - \overline{z} = 2i \operatorname{Im} z$;

d) $z \cdot \overline{z} = \operatorname{Re}^2 z + \operatorname{Im}^2 z = |z|^2$ *(for the definition of $|z|$ see Definition 21.3 below);*

e) $\frac{1}{z} = \frac{\overline{z}}{\operatorname{Re}^2 z + \operatorname{Im}^2 z} = \frac{\overline{z}}{|z|^2}$, $z \neq 0$;

f) $\overline{\left(\frac{z_1}{z_2}\right)} = \overline{z_1}/\overline{z_2}$, $z_2 \neq 0$;

g) $\overline{\overline{z}} = z$.

Proof. Equalities a), b), c), d), f) and g) immediately follow from the definition of complex conjugate and (42), (43), (44). Equality e) follows from d). Indeed, multiplying the nominator and denominator of $\frac{1}{z}$ by $\overline{z}$ and using d), we have

$$\frac{1}{z} = \frac{\overline{z}}{z \cdot \overline{z}} = \frac{\overline{z}}{\operatorname{Re}^2 z + \operatorname{Im}^2 z} = \frac{\overline{z}}{|z|^2}.$$

□

Exercise 21.4. Prove equalities a)-d), f) and g).

Definition 21.3. The number $|z| = \sqrt{x^2 + y^2} = \sqrt{\operatorname{Re}^2 z + \operatorname{Im}^2 z}$ is called the **absolute value** of a complex number $z = x + yi \in \mathbb{C}$.

Theorem 21.2. *Let $z_1, z_2, z \in \mathbb{C}$. Then the absolute value satisfies the following properties:*

a) $|z| = \sqrt{z \cdot \overline{z}}$;

b) $|z| > 0$ *unless* $z = 0$;

c) $|z| = |\overline{z}|$;

d) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$;

e) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$;

f) $|\operatorname{Re} z| \le |z|, \quad |\operatorname{Im} z| \le |z|$;

g) $|z_1 + z_2| \le |z_1| + |z_2|$.

Proof. Properties a), b), c) and f) easily follow from Definition 21.3. To show d), we compute for $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$

$$\begin{aligned}|z_1 \cdot z_2|^2 &= \operatorname{Re}^2(z_1 \cdot z_2) + \operatorname{Im}^2(z_1 \cdot z_2) \overset{(43)}{=} (x_1x_2 - y_1y_2)^2 + (y_1x_2 + x_1y_2)^2 \\ &= x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2 + y_1^2x_2^2 + 2x_1x_2y_1y_2 + x_1^2y_2^2 \\ &= x_1^2x_2^2 + y_1^2y_2^2 + y_1^2x_2^2 + x_1^2y_2^2 = (x_1^2 + y_1^2)(x_2^2 + y_2^2) = |z_1|^2 \cdot |z_2|^2.\end{aligned}$$

Thus, d) holds. Next, we prove e). So, for $z_2 \neq 0$ we have

$$\left|\frac{z_1}{z_2}\right| \overset{\text{Thm 21.1 e)}}{=} \left|\frac{z_1 \cdot \overline{z_2}}{|z_2|^2}\right| \overset{\text{d)}}{=} \frac{|z_1| \cdot |\overline{z_2}|}{|z_2|^2} \overset{\text{c)}}{=} \frac{|z_1|}{|z_2|}.$$

Now, we check triangle inequality g):

$$\begin{aligned}|z_1 + z_2|^2 &\overset{\text{a)}}{=} (z_1 + z_2) \cdot \overline{(z_1 + z_2)} \overset{\text{Thm 21.1 a)}}{=} (z_1 + z_2) \cdot (\overline{z_1} + \overline{z_2}) = z_1 \cdot \overline{z_1} + z_1 \cdot \overline{z_2} + z_2 \cdot \overline{z_1} + z_2 \cdot \overline{z_2} \\ &\overset{\text{a) \& Thm 21.1 g)}}{=} |z_1|^2 + z_1 \cdot \overline{z_2} + \overline{\overline{z_2 \cdot \overline{z_1}}} + |z_2|^2 \overset{\text{Thm 21.1 b), g)}}{=} |z_1|^2 + z_1 \cdot \overline{z_2} + \overline{\overline{z_2} \cdot z_1} + |z_2|^2 \\ &\overset{\text{Thm 21.1 c)}}{=} |z_1|^2 + 2\operatorname{Re}(z_1 \cdot \overline{z_2}) + |z_2|^2 \overset{\text{f)}}{\le} |z_1|^2 + 2|z_1 \cdot \overline{z_2}| + |z_2|^2 \\ &\overset{\text{c), d)}}{=} |z_1|^2 + 2|z_1| \cdot |z_2| + |z_2|^2 = (|z_1| + |z_2|)^2.\end{aligned}$$

□

Exercise 21.5. Let $z, w \in \mathbb{C}$. Prove the parallelogram law $|z - w|^2 + |z + w|^2 = 2(|z|^2 + |w|^2)$.

Exercise 21.6. Let $z, w \in \mathbb{C}$ with $\overline{z}w \neq 1$ such that either $|z| = 1$ or $|w| = 1$. Prove that

$$\left|\frac{z - w}{1 - \overline{z}w}\right| = 1.$$

Exercise 21.7. Let z be a complex number with $|z| < \frac{1}{2}$. Show that

$$|(1 + i)z^2 + iz| < \frac{3}{4}.$$

Exercise 21.8. Solve the following equations:
a) $|z| - z = 1 + 2i$; b) $|z| + z = 2 + i$.

21.3 Complex Plane and Polar form of complex numbers

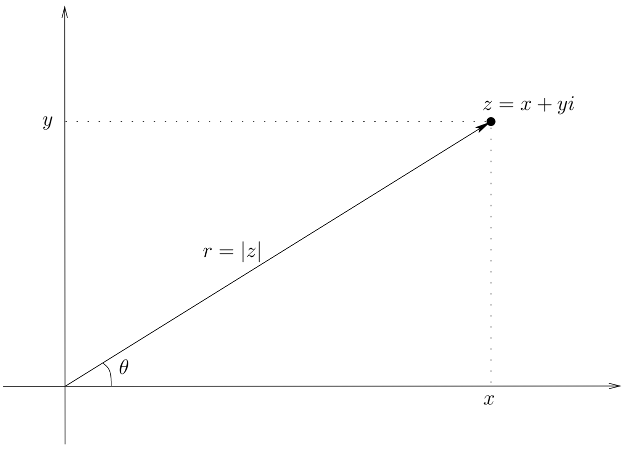
In this section, we will identify complex numbers with points of a plane which we will call the **complex plane**. So, we will identify a number $z = x + iy \in \mathbb{C}$ with the point (x, y) of $\mathbb{R}^2$. The point (x, y) is called the **rectangular coordinates** of z. We will also identify z with its polar coordinates (r, θ), where r is the length of the vector $\overline{(0,0),(x,y)}$ and equals the absolute volume of z, and θ is the angle between the positive real axis and the vector $(0,0),(x,y)$. The angle θ is called the **argument** of z and is denoted by $\theta = \arg z$. We remark that for $z \neq 0$ the argument θ is uniquely determined up to integer multiples of 2π.

By the definition of sin and cos, it is easy to see that

$$\cos\theta = \frac{x}{r} \quad \text{and} \quad \sin\theta = \frac{y}{r}.$$

Consequently, we can write the number $z = x + yi$ in the form

$$z = r(\cos\theta + i\sin\theta),$$



where $r = |z|$ and $\theta = \arg z$. This form is called the **polar form** of the complex number z.

Example 21.3. Let us write the number $z = 1 + i$ in the polar form. For this we compute $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. The argument θ can be found from the equalities $\cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$. Thus, $\theta = \frac{\pi}{4}$. Hence, $z = 1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.

Exercise 21.9. Write the following complex numbers in the polar form:
a) i; b) $1 - i$; c) $-1 + \sqrt{3}i$; d) $-2 - 2i$.

It turns out, that the polar form of complex numbers is convenient for the multiplication and division.

Theorem 21.3. *Let* $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ *and* $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ *be complex numbers, written in the polar form. Then*

$$z_1 \cdot z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)), \tag{45}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)), \quad z_2 \neq 0. \tag{46}$$

Proof. The equalities immediately follows from (43), (44) and the formulas

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2,$$
$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2.$$

□

Remark 21.4. Setting

$$e^{i\theta} = \cos\theta + i\sin\theta, \text{[1]}$$

equalities (45) and (46) can be rewritten as follows

$$z_1 \cdot z_2 = \left(r_1 e^{i\theta_1}\right) \cdot \left(r_2 e^{i\theta_2}\right) = r_1 r_2 e^{i(\theta_1 + \theta_2)},$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}, \quad z_2 \neq 0.$$

[1] This formula is called Euler's formula and can be obtain from the Taylor expansion of functions of complex argument.

Exercise 21.10. Simplify the expression $\frac{\cos\theta+i\sin\theta}{\cos\varphi-i\sin\varphi}$.

Exercise 21.11. Compute $\frac{(1-\sqrt{3}i)(\cos\theta+i\sin\theta)}{2(1-i)(\cos\theta-i\sin\theta)}$.

Corollary 21.1 (De Moivre's formula)**.** *Let $z = r(\cos\theta + i\sin\theta) \neq 0$ be a complex number. Then for each $n \in \mathbb{Z}$*

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

Proof. The corollary immediately follows from Theorem 21.3. □

Exercise 21.12. Compute: a) $(1+i)^{25}$; b) $(\sqrt{3}-3i)^{15}$; c) $\left(\frac{1+\sqrt{3}i}{1-i}\right)^{20}$; d) $\left(1-\frac{\sqrt{3}-i}{2}\right)^{24}$.

21.4 Roots of Complex Numbers

Let $n \in \mathbb{N}$ be fixed.

Definition 21.4. A complex number w is called an **n-th root** of $z \in \mathbb{C}$ if $w^n = z$.

Theorem 21.4. *Let $z = r(\cos\theta + i\sin\theta) \neq 0$ be a complex number. Then z has n different n-th roots given by the formula*

$$w_k = \sqrt[n]{r}\left(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\right), \quad k = 0, 1, \ldots, n-1,$$

where $\sqrt[n]{r}$ is the usual n-th root of the positive real number r.

Proof. Let $w = \rho(\cos\varphi + i\sin\varphi)$ be a complex number written in the polar form such that $w^n = z$. Then

$$w^n = \rho^n(\cos n\varphi + i\sin n\varphi) = r(\cos\theta + i\sin\theta),$$

by Corollary 21.1. Thus, $\rho^n = r$ and $n\varphi = \theta + 2\pi k$, $k \in \mathbb{Z}$. This implies that $\rho = \sqrt[n]{r}$ and $\varphi = \frac{\theta+2\pi k}{n}$, $k \in \mathbb{Z}$. So, we obtain that the numbers

$$w_k = \sqrt[n]{r}\left(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\right), \quad k \in \mathbb{Z},$$

are n-th roots of z. By the periodicity of sin and cos, one can see that there are only n different w_k, $k = 0, \ldots, n-1$. □

Example 21.4. Let us compute 4-th root of $z = -1$. First we write -1 in the polar form: $-1 = 1(\cos\pi + i\sin\pi)$. Then $w_k = \cos\frac{\pi+2\pi k}{4} + i\sin\frac{\pi+2\pi k}{4}$, $k = 0, 1, 2, 3$, are 4-th roots of -1, by Theorem 21.4. Thus, $w_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, $w_1 = \cos\frac{\pi+2\pi}{4} + i\sin\frac{\pi+2\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, $w_2 = \cos\frac{\pi+4\pi}{4} + i\sin\frac{\pi+4\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$, $w_3 = \cos\frac{\pi+6\pi}{4} + i\sin\frac{\pi+6\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

Remark 21.5. The n-th roots of $z \neq 0$ form a regular n-gon in the complex plane with center 0. The vertices of this n-gon lie on the circle with center 0 and the radius $\sqrt[n]{|z|}$.

Exercise 21.13. Solve the following equations:
a) $z^5 - 2 = 0$; b) $z^4 + i = 0$; c) $z^3 - 4i = 0$.

Exercise 21.14. Compute a) 6-th roots of $\frac{1-i}{\sqrt{3}+i}$; b) 8-th roots of $\frac{1+i}{\sqrt{3}-i}$.