20 Lecture 20 – Series with Arbitrary Terms

20.1 Root and Ratio Tests for Series with Positive Terms

Theorem 20.1 (Ratio Test). *Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n > 0$, $n \geq 1$, and let there exist a limit $r := \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if $r < 1$ and diverges if $r > 1$.*

Proof. Let $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1$. We take $q \in (r, 1)$. Then there exists $N \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} < q = \frac{q^{n+1}}{q^n}$ for all $n \geq N$. Thus, using Theorem 19.6 (iii) and the convergence of the geometric series for $|q| < 1$ (see Example 19.2), we have that the series $a_N + a_{N+1} + \ldots = \sum_{n=N}^{\infty} a_n$ converges and, hence, $\sum_{n=1}^{\infty} a_n$ also converges.

If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r > 1$, then there exists $N \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} > 1$ for all $n \geq N$. Consequently, $a_n < a_{n+1}$ for all $n \geq N$. So, we obtain that $0 < a_N < a_{N+1} < a_{N+2} < \ldots$. This implies that $a_n \not\to 0$, $n \to \infty$. Hence, the series $\sum_{n=1}^{\infty} a_n$ diverges, according to Theorem 19.1. □

Example 20.1. The series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges for all $x > 0$. Indeed,

$$r = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \lim_{n\to\infty} \frac{x}{n+1} = 0 < 1.$$

Exercise 20.1. Prove that the following series converge:
a) $\sum_{n=1}^{\infty} \frac{3^n (n!)^2}{(2n)!}$; b) $\sum_{n=1}^{\infty} \frac{7^n (n!)^2}{n^{2n}}$.

Theorem 20.2 (Root Test). *Let $\sum_{n=1}^{\infty} a_n$ be a series with $a_n \geq 0$, $n \geq 1$, and let $r := \overline{\lim_{n\to\infty}} \sqrt[n]{a_n}$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if $r < 1$ and diverges if $r > 1$.*

Proof. Let $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} = r < 1$ and let q be a number from $(r, 1)$. Then there exists $N \in \mathbb{N}$ such that $\sqrt[n]{a_n} < q$ for all $n \geq N$. So, $a_n < q^n$ for all $n \geq N$. By Theorem 19.6 (i), the series $\sum_{n=N}^{\infty} a_n$ converges due to the convergence of the geometric series $\sum_{n=1}^{\infty} q^n$ for $|q| < 1$.

If $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} = r > 1$, then there exists a subsequence $(\sqrt[n_k]{a_{n_k}})_{k\geq 1}$ such that $\sqrt[n_k]{a_{n_k}} \to r$, $k \to \infty$, since the upper limit is also a subsequential limit (see Theorem 5.1). Hence, there exists $K \in \mathbb{N}$ such that $\sqrt[n_k]{a_{n_k}} > 1$ for all $k \geq K$. Consequently, $a_{n_k} > 1$ for all $k \geq K$. This implies that $a_n \not\to 0$, $n \to \infty$, since the sequence $(a_n)_{n\geq 1}$ has an subsequence which does not converge to 0. □

Example 20.2. The series $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ converges, since $r = \lim_{n\to\infty} \sqrt[n]{\frac{n^3}{2^n}} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^3}{2} = \frac{1}{2} < 1$.

Exercise 20.2. Prove that the following series converge:
a) $\sum_{n=1}^{\infty} \frac{3^n}{(\ln n)^n}$; b) $\sum_{n=1}^{\infty} \frac{n^{n^2} 2^n}{(n+1)^{n^2}}$.

20.2 Series with Arbitrary Terms

20.2.1 Absolute and Conditional Convergence

Definition 20.1. A series

$$a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n \tag{37}$$

is said to be **absolutely convergent**, if the series

$$|a_1| + |a_2| + \ldots + |a_n| + \ldots = \sum_{n=1}^{\infty} |a_n| \tag{38}$$

converges. If series (38) diverges but (37) converges, then series (37) is called **conditionally convergent**.

Theorem 20.3. *If a series* $\sum\limits_{n=1}^{\infty} a_n$ *absolutely converges, then it converges and*

$$\left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty} |a_n|.$$

Proof. We note that terms of the series

$$\sum_{n=1}^{\infty} (a_n + |a_n|) \tag{39}$$

satisfy the property $0 \le a_n + |a_n| \le 2|a_n|$, $n \ge 1$. Thus, series (39) converges due to the convergence of the series $\sum_{n=1}^{\infty} 2|a_n|$ and Theorem 19.6 (i). Summing series (39) with the series $\sum\limits_{n=1}^{\infty} (-|a_n|)$, which also converges, we have that the series $\sum\limits_{n=1}^{\infty} (a_n + |a_n| - |a_n|) = \sum\limits_{n=1}^{\infty} a_n$ converges, by Theorem 19.2. □

We set $a_n^+ := \max\{a_n, 0\}$ and $a_n^- := -\min\{a_n, 0\}$, $n \ge 1$. Then $a_n = a_n^+ - a_n^-$ and $|a_n| = a_n^+ + a_n^-$ for all $n \ge 1$.

Theorem 20.4. *A series* $\sum\limits_{n=1}^{\infty} a_n$ *absolutely converges iff the series* $\sum\limits_{n=1}^{\infty} a_n^+$ *and* $\sum\limits_{n=1}^{\infty} a_n^-$ *converge. Moreover,*

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-, \quad \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} a_n^+ + \sum_{n=1}^{\infty} a_n^-.$$

Exercise 20.3. Prove Theorem 20.4. (*Hint:* Use the equalities $0 \le a_n^+ \le |a_n|$ and $0 \le a_n^- \le |a_n|$)

Corollary 20.1. *Let a series* $\sum\limits_{n=1}^{\infty} a_n$ *conditionally converge. Then the series* $\sum\limits_{n=1}^{\infty} a_n^+$ *and* $\sum\limits_{n=1}^{\infty} a_n^-$ *diverge.*

Proof. We assume that $\sum\limits_{n=1}^{\infty} a_n^+$ converges. Using Theorem 19.2, we obtain that the series $\sum\limits_{n=1}^{\infty} a_n^+ - \sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (a_n^+ - a_n) = \sum\limits_{n=1}^{\infty} a_n^-$ also converges. But then, by Theorem 20.4, the series $\sum\limits_{n=1}^{\infty} a_n$ absolutely converges that contradicts the assumption of the corollary. □

Exercise 20.4. Show that the following series absolutely converge:
a) $\sum\limits_{n=1}^{\infty} \frac{\sin n}{n^2}$; b) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n n!}{(2n)!}$.

20.2.2 Dirichlet's and Abel's Tests

Theorem 20.5 (Dirichlet's test). *Let sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ satisfy the following properties:*

1) $(a_n)_{n\geq 1}$ is a monotone sequence;

2) $a_n \to 0$, $n \to \infty$;

3) there exists $C > 0$ such that $\left|\sum\limits_{k=1}^{n} b_n\right| \leq C$ for all $n \geq 1$.

Then the series $\sum\limits_{n=1}^{\infty} a_n b_n$ converges.

Proof. For proof of the theorem see Theorem 3.42 [2]. □

Example 20.3. The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

conditionally converges. Indeed, taking $a_n := \frac{1}{n}$ and $b_n := (-1)^{n+1}$, $n \geq 1$, we can see that the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ satisfy the conditions of Dirichlet's test (condition 3) is satisfied with $C = 1$). Thus, the series $\sum\limits_{n=1}^{\infty} a_n b_n = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. But the series $\sum\limits_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges (see Example 19.3).

Example 20.4. The series $\sum\limits_{n=1}^{\infty} \frac{\sin n}{n}$ converges. To prove this, we take $a_n := \frac{1}{n}$, $b_n := \sin n$, $n \geq 1$. The sequence $(a_n)_{n\geq 1}$ is monotone and converges to 0. Next, we compute for $n \geq 1$

$$\begin{aligned}\sum_{k=1}^{n} \sin k &= \frac{1}{\sin\frac{1}{2}} \sum_{k=1}^{n} \sin k \cdot \sin\frac{1}{2} = \frac{1}{2\sin\frac{1}{2}} \sum_{k=1}^{n} \left(\cos\left(k - \frac{1}{2}\right) - \cos\left(k + \frac{1}{2}\right)\right) \\ &= \frac{1}{2\sin\frac{1}{2}} \left(\cos\frac{1}{2} - \cos\left(n + \frac{1}{2}\right)\right).\end{aligned}$$

Hence,

$$\left|\sum_{k=1}^{n} \sin k\right| = \left|\frac{1}{2\sin\frac{1}{2}} \left(\cos\frac{1}{2} - \cos\left(n + \frac{1}{2}\right)\right)\right| \leq \frac{1}{\sin\frac{1}{2}}, \quad n \geq 1,$$

and, consequently, condition 3) of Dirichlet's test is satisfied. Hence, the series $\sum\limits_{n=1}^{\infty} \frac{\sin n}{n}$ converges.

Exercise 20.5. Show that the series $\sum\limits_{n=1}^{\infty} \frac{|\sin n|}{n}$ diverges. (*Hint:* Use the equality $|\sin a| \geq \sin^2 a = \frac{1-\cos 2a}{2}$ and then show that the series $\sum\limits_{n=1}^{\infty} \frac{1}{2n}$ diverges and $\sum\limits_{n=1}^{\infty} \frac{\cos 2n}{2n}$ converges).

Exercise 20.6. Prove the convergence of the following sequences:
a) $\sum_{n=1}^{\infty}(-1)^{\frac{n(n+1)}{2}}\frac{1}{\sqrt{n}}$; b) $\sum_{n=1}^{\infty}\frac{\sin 3n}{\sqrt{n}}$; c) $\sum_{n=1}^{\infty}\frac{\cos n}{n}$.

Corollary 20.2 (Leibniz's test)**.** *Let a sequence $(a_n)_{n\geq 1}$ satisfy the following properties:*

1) $0 \leq a_{n+1} \leq a_n$ for $n \geq 1$;

2) $a_n \to 0$, $n \to \infty$.

Then the series

$$a_1 - a_2 + a_3 - a_4 + \ldots = \sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

converges.

Proof. The corollary follows from Dirichlet's test taking $b_n := (-1)^{n+1}$, $n \geq 1$. □

Example 20.5. The series $\sum_{n=1}^{\infty}(-1)^n \ln\frac{n+1}{n}$ converges due to Leibniz's test, since the sequence $(a_n)_{n\geq 1} = \left(\ln\frac{n+1}{n}\right)_{n\geq 1}$ decreases to 0. Indeed, $a_n = \ln\frac{n+1}{n} = \ln\left(1+\frac{1}{n}\right) > \ln\left(1+\frac{1}{n+1}\right) = a_{n+1} > 0$ because $1+\frac{1}{n} > 1+\frac{1}{n+1}$ and ln is an increasing function.

Theorem 20.6 (Abel's test)**.** *Let sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ satisfy the following properties:*

1) $(a_n)_{n\geq 1}$ is monotone;

2) $(a_n)_{n\geq 1}$ is bounded;

3) the series $\sum_{n=1}^{\infty} b_n$ converges.

Then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

Proof. In order to prove Abel's test, we are going to use Dirichlet's test. Since the sequence $(a_n)_{n\geq 1}$ is monotone and bounded, it has a limit $a \in \mathbb{R}$, by Theorem 4.1. Applying Dirichlet's test to the sequences $(a_n - a)_{n\geq 1}$ and $(b_n)_{n\geq 1}$, we get that the series $\sum_{n=1}^{\infty}(a_n - a)b_n$ convergence. Thus, the series $\sum_{n=1}^{\infty}(a_n - a)b_n + a\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n b_n$ is convergent due to the convergence of $\sum_{n=1}^{\infty} b_n$ and Theorem 19.2. □

Exercise 20.7. Prove the convergence of the series $\sum_{n=1}^{\infty}(-1)^n\frac{\arctan n}{\sqrt{n}}$.

20.2.3 Permutation of Terms of a Series

Definition 20.2. A bijection $\sigma : \mathbb{N} \to \mathbb{N}$ is called a **permutation**.

In this section, we will study series obtained from permutation of their terms, i.e.

$$a_{\sigma(1)} + a_{\sigma(2)} + \ldots + a_{\sigma(n)} + \ldots = \sum_{n=1}^{\infty} a_{\sigma(n)}. \tag{40}$$

According to Example 20.3, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. Moreover, one can show that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$$

But it turns out that a rearrangement of the series gives other finite sum, e.g.

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots = \frac{3}{2} \ln 2.$$

So, we see that there exist series whose sums depend on order of their terms.

Theorem 20.7. *Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series. Then for every permutation σ the permuted series* (40) *converges to the same sum, i.e.*

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \sum_{n=1}^{\infty} a_n.$$

Proof. For proof of the theorem see Theorem 3.55 [2]. □

Theorem 20.8 (Riemann rearrangement theorem)**.** *Let $\sum_{n=1}^{\infty} a_n$ be conditionally convergent and $s \in \mathbb{R} \cup \{-\infty, +\infty\}$. Then there exists a permutation σ such that*

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = s.$$

Proof. For proof of the theorem in more general setting see Theorem 3.54 [2]. □

References

[1] K.A. Ross. *Elementary Analysis: The Theory of Calculus*. Undergraduate Texts in Mathematics. Springer New York, 2013.

[2] Walter Rudin. *Principles of mathematical analysis*. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, third edition, 1976. International Series in Pure and Applied Mathematics.