19 Lecture 19 – Series

19.1 Definition and Elementary Properties of Series

Let $(a_n)_{n\geq 1}$ be a sequence of the real numbers. For each $n \in \mathbb{N}$ we set

$$s_n := a_1 + a_2 + \ldots + a_n.$$

Definition 19.1. The sequence $(s_n)_{n\geq 1}$ is called a **series** and is denoted by

$$a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n. \tag{35}$$

Elements of the sequence $(s_n)_{n\geq 1}$ are called the **partial sums of series** (35). If the sequence $(s_n)_{n\geq 1}$ converges to a real number s, then series (35) is said to be **convergent**, and the number s is called the **sum of series** (35) and is denoted by

$$s = \sum_{n=1}^{\infty} a_n.$$

If the sequence $(s_n)_{n\geq 1}$ has no a finite limit, then series (35) is said to be **divergent**.

Theorem 19.1. *If a series* $\sum\limits_{n=1}^{\infty} a_n$ *converges, then* $a_n \to 0$, $n \to \infty$.

Proof. Indeed, since $a_n = s_n - s_{n-1}$ for all $n \geq 2$, we have $a_n = s_n - s_{n-1} \to s - s = 0$, $n \to \infty$. □

Exercise 19.1. Prove that the convergence of a series $\sum\limits_{n=1}^{\infty} a_n$ implies that $a_n + a_{n+1} + \ldots + a_{2n} \to 0$, $n \to \infty$.

Example 19.1. The series

$$1 + 1 + \ldots + 1 + \ldots$$

and

$$1 - 1 + 1 - 1 + \ldots + (-1)^{n+1} + \ldots$$

diverge, since their terms $a_n = 1$, $n \geq 1$, for the first series and $a_n = (-1)^{n+1}$, $n \geq 1$, for the second one do not converge to 0.

Example 19.2 (Geometric series). For $q \in \mathbb{R}$ the series

$$1 + q + q^2 + \ldots + q^n + \ldots = \sum_{n=1}^{\infty} q^{n-1} = \sum_{n=0}^{\infty} q^n \tag{36}$$

is called the **geometric series**. Its partial sums $s_n = 1 + q + q^2 + \ldots + q^{n-1}$ are equal to $\frac{1-q^n}{1-q}$ for all $n \geq 1$. Thus, series (36) converges and

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q},$$

for $|q| < 1$. If $|q| \geq 1$, then the geometric series diverges.

Example 19.3 (Harmonic series)**.** The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$$

diverges. In order to prove this, we assume that the series converges and its sum equal s. Then $s_{2n} - s_n \to s - s = 0$, $n \to \infty$. But for each $n \geq 1$

$$s_{2n} - s_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n} \geq n \frac{1}{2n} = \frac{1}{2}$$

that contradicts the convergence of $(s_{2n} - s_n)_{n \geq 1}$ to 0.

Exercise 19.2. Show that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)} + \ldots = 1$.

Theorem 19.2. *Let series* $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ *converge and* $c \in \mathbb{R}$. *Then the series* $\sum_{n=1}^{\infty} c a_n$, $\sum_{n=1}^{\infty} (a_n + b_n)$ *also converge and* $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

Proof. The proof of the statement immediately follows from Definition 19.1 and Theorem 3.8. Indeed,

$$\sum_{n=1}^{\infty} c a_n = \lim_{n\to\infty} \sum_{k=1}^{n} c a_n = c \lim_{n\to\infty} \sum_{k=1}^{n} a_n = c \sum_{n=1}^{\infty} a_n$$

and

$$\sum_{n=1}^{\infty} (a_n + b_n) = \lim_{n\to\infty} \sum_{k=1}^{n} (a_k + b_k) = \lim_{n\to\infty} \sum_{k=1}^{n} a_k + \lim_{n\to\infty} \sum_{k=1}^{n} b_k = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

□

Theorem 19.3 (Cauchy criterion)**.** *A series* $\sum_{n=1}^{\infty} a_n$ *converges iff*

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad \forall p \in \mathbb{N} : \quad |a_{n+1} + a_{n+2} + \ldots + a_{n+p}| < \varepsilon.$$

Proof. We remark that $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums $(s_n)_{n\geq 1}$ converges. Thus, using Theorem 5.3, we have that the convergence of $(s_n)_{n\geq 1}$ is equivalent to

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad \forall p \in \mathbb{N} : \quad |s_{n+p} - s_n| < \varepsilon.$$

Hence, the statement follows from the equality $s_{n+p} - s_n = a_{n+1} + a_{n+2} + \ldots + a_{n+p}$. □

19.2 Series with Positive Terms

Theorem 19.4. *Let terms of a series* $\sum_{n=1}^{\infty} a_n$ *are non-negative, that is,* $a_n \geq 0$ *for all* $n \geq 1$. *The series* $\sum_{n=1}^{\infty} a_n$ *converges iff the sequence of its partial sums* $(s_n)_{n\geq 1}$ *is bounded.*

Proof. We note that the sequence $(s_n)_{n\geq 1}$ increases. Hence, the statement follows from theorems 4.1 and 3.5. □

Theorem 19.5 (Integral criterion for convergence)**.** *Let $f : [1, +\infty) \to \mathbb{R}$ be a non-negative decreasing function and $f(n) = a_n$ for all $n \geq 1$. Then the convergence of the series $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} f(n)$ is equivalent to the convergence of the improper integral $\int_1^{+\infty} f(x)dx$.*

Proof. Using the monotonicity of the function f and Corollary 16.1, we can estimate for each $n \geq 2$

$$a_n = f(n) \leq \int_{n-1}^{n} f(x)dx \leq f(n-1) = a_{n-1}.$$

So, if the improper integral $\int_1^{+\infty} f(x)dx$ converges, then for every $n \geq 1$

$$s_n = \sum_{k=1}^{n} a_k = a_1 + \sum_{k=2}^{n} a_k = a_1 + \sum_{k=2}^{n} \int_{k-1}^{k} f(x)dx = a_1 + \int_1^n f(x)dx \leq a_1 + \int_1^{+\infty} f(x)dx.$$

Hence, the sequence $(s_n)_{n\geq 1}$ is bounded and, consequently, the series $\sum\limits_{n=1}^{\infty} a_n$ converges, by Theorem 19.4.

Next, if the series $\sum\limits_{n=1}^{\infty} a_n$ converges, then for each $z > 1$

$$\varphi(z) = \int_1^z f(x)dx \leq \int_1^n f(x)dx = \sum_{k=2}^{n} \int_{k-1}^{k} f(x)dx \leq \sum_{k=2}^{n} a_{k-1} = \sum_{k=1}^{n-1} a_k \leq \sum_{k=1}^{\infty} a_k =: C,$$

where $n := \lfloor z \rfloor + 1$. Thus, the integral $\int_1^{+\infty} f(x)dx$ converges, by Theorem 18.2. □

Example 19.4. The series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$, $p > 0$, converges for $p > 1$ and diverges for $p \leq 1$. This follows from Theorem 19.5 and the fact that the integral $\int_1^{+\infty} \frac{dx}{x^p}$ converges for $p > 1$ and diverges for $p \leq 1$ (see Example 18.3).

Exercise 19.3. Show that the series $\sum\limits_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

Theorem 19.6 (Comparison criterion)**.** *Let $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ be series.*

(i) If $0 \leq a_n \leq b_n$, $n \geq 1$, then the convergence of $\sum\limits_{n=1}^{\infty} b_n$ implies the convergence of $\sum\limits_{n=1}^{\infty} a_n$.

(ii) Let $a_n > 0$, $b_n > 0$, $n \geq 1$, and there exists a limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = C, \quad 0 \leq C \leq +\infty.$$

If $C < +\infty$, then the convergence of $\sum\limits_{n=1}^{\infty} b_n$ implies the convergence of $\sum\limits_{n=1}^{\infty} a_n$. If $C > 0$, then the divergence of $\sum\limits_{n=1}^{\infty} b_n$ implies the divergence of $\sum\limits_{n=1}^{\infty} a_n$. Consequently, the convergences of both series are equivalent in the case $0 < C < +\infty$.

(iii) If $a_n > 0$, $b_n > 0$ and $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for all $n \geq 1$, then the convergence of $\sum_{n=1}^{\infty} b_n$ implies the convergence of $\sum_{n=1}^{\infty} a_n$.

Proof. We prove only (i). We estimate for each $n \geq 1$

$$0 \leq s_n = \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k \leq \lim_{n\to\infty} \sum_{k=1}^{n} b_k = \sum_{k=1}^{+\infty} b_k.$$

Thus, the sequence $(s_n)_{n\geq 1}$ is bounded that implies the convergence of the series $\sum_{n=1}^{\infty} a_n$, according to Theorem 19.4. □

Exercise 19.4. Prove Theorem 19.6 (ii), (iii). (*Hint:* To prove (iii), note that $\frac{a_{n+1}}{b_{n+1}} \leq \frac{a_n}{b_n} \leq \ldots \frac{a_1}{b_1}$)

Remark 19.1. We will write, $a_n \sim b_n$, $n \to \infty$, if $\frac{a_n}{b_n} \to 1$, $n \to \infty$. So, Theorem 19.6 (ii) implies that the convergence of $\sum_{n=1}^{\infty} a_n$ is equivalent to the convergence of $\sum_{n=1}^{\infty} b_n$, if $a_n \sim b_n$, $n \to \infty$.

Example 19.5. The series $\sum_{n=1}^{\infty} n \sin \frac{1}{n^3}$ converges, since $n \sin \frac{1}{n^3} \sim \frac{n}{n^3} = \frac{1}{n^2}$, $n \to \infty$, and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (see Example 19.4).

Exercise 19.5. Prove the convergence of the following series:
a) $\sum_{n=1}^{\infty} \frac{n+1}{n^3}$; b) $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$; c) $\sum_{n=1}^{\infty} \left(1 - \cos \frac{1}{n}\right)$; d) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n(n+1)}$; e) $\sum_{n=1}^{\infty} \left(\sqrt{n^2+1} - n\right)^2$; f) $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$;
g) $\sum_{n=1}^{\infty} \frac{n^{n-2}}{e^n n!}$; h) $\sum_{n=2}^{\infty} \left(\ln \frac{n}{n-1} - \frac{1}{n}\right)$.