13 Lecture 13 – L'Hospital's Rule and Taylor's Theorem

13.1 L'Hospital's Rule

Theorem 13.1 (L'Hospital's Rule). *Let $a \in \mathbb{R}$ or $a = -\infty$ and functions $f, g : (a, b) \to \mathbb{R}$ satisfy the following properties*

1) *f, g are differentiable on (a, b);*

2) *$\lim\limits_{x\to a+} f(x) = \lim\limits_{x\to a+} g(x) = 0$ or $\lim\limits_{x\to a+} |g(x)| = +\infty$;*

3) *$g'(x) \neq 0$ for all $x \in (a, b)$;*

4) *there exists $\lim\limits_{x\to a+} \frac{f'(x)}{g'(x)} =: L \in \mathbb{R}$.*

Then there exists $\lim\limits_{x\to a+} \frac{f(x)}{g(x)} = L$.

Proof. We will only give a proof for the case $a \in \mathbb{R}$ and $\lim\limits_{x\to a+} f(x) = \lim\limits_{x\to a+} g(x) = 0$. For the general case see e.g. [1, p.242-244].

We first extend the functions f and g to the interval $[a, b)$, setting $f(a) = g(a) := 0$. According to assumption 2), f and g are continuous at the point a. Since f, g are differentiable on (a, b), they are continuous also at each point of (a, b), by Theorem 10.2. Thus, f, g are continuous on $[a, b)$. Next, we note that $g(x) \neq 0$ for all $x \in (a, b)$. Indeed, if $g(x_0) = 0$ for some $x_0 \in (a, b)$, then applying Rolle's theorem (see Theorem 11.3) to the function $g : [a, x_0] \to \mathbb{R}$, we obtain that there exists $c \in (a, x_0)$ such that $g'(c) = 0$, that is impossible by assumption 3).

Next, to show that $\lim\limits_{x\to a+} \frac{f(x)}{g(x)} = L$, we are going to use Theorem 7.7. Let $\varepsilon > 0$ be fixed. By Theorem 7.7 and assumption 4),

$$\exists \delta > 0 \ \ \forall x \in (a, a + \delta) : \ \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon.$$

Applying the Cauchy theorem (see Theorem 11.5) to the functions $f, g : [a, x] \to \mathbb{R}$, we have for all $x \in (a, a + \delta)$

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f(x) - f(a)}{g(x) - g(a)} - L \right| = \left| \frac{f'(c)}{g'(c)} - L \right| < \varepsilon,$$

where $c \in (a, x) \subset (a, a + \delta)$. □

Remark 13.1. A similar statement is true for the left-sided limit as x goes to b.

Example 13.1. Using L'Hospital's Rule, we compute the following limits:

a) $\lim\limits_{x\to 0} \frac{\sin x}{x} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0} \frac{(\sin x)'}{(x)'} = \lim\limits_{x\to 0} \frac{\cos x}{1} = 1$;

b) $\lim\limits_{x\to 0} x \ln x \overset{0\cdot\infty}{=} \lim\limits_{x\to 0} \frac{\ln x}{\frac{1}{x}} \overset{\frac{\infty}{\infty}}{=} \lim\limits_{x\to 0} \frac{(\ln x)'}{\left(\frac{1}{x}\right)'} = \lim\limits_{x\to 0} \frac{\frac{1}{x}}{\left(-\frac{1}{x^2}\right)} = -\lim\limits_{x\to 0} x = 0$;

c) $\lim\limits_{x\to 0} (\cos x)^{\frac{1}{x^2}} \overset{1^\infty}{=} \lim\limits_{x\to 0} e^{\frac{1}{x^2} \ln \cos x}$.

We compute

$$\lim_{x\to 0} \frac{1}{x^2} \ln \cos x = \lim_{x\to 0} \frac{\ln \cos x}{x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{(\ln \cos x)'}{(x^2)'} = \lim_{x\to 0} \frac{-\frac{\sin x}{\cos x}}{2x} = -\frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x \cos x} = -\frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x} = -\frac{1}{2}.$$

Thus, by the continuity of the function $f(x) = e^x$, $x \in \mathbb{R}$, we have
$\lim\limits_{x\to 0} e^{\frac{1}{x^2}\ln\cos x} = e^{\lim\limits_{x\to 0}\frac{1}{x^2}\ln\cos x} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$.

See [1, p.245-248] for more examples of the application of L'Hospital's Rule.

Exercise 13.1. Using L'Hospital's Rule, show that
a) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2} = 1$; b) $\lim\limits_{x\to 0}\frac{\ln(1+x)}{\sin x} = 1$; c) $\lim\limits_{x\to e}\frac{(\ln x)^\alpha - \left(\frac{x}{e}\right)^\beta}{x-e} = \frac{\alpha-\beta}{e}$, where α, β are some real numbers;
d) $\lim\limits_{x\to 1}\frac{\left(\frac{4}{\pi}\arctan x\right)^\alpha - 1}{\ln x} = \frac{2\alpha}{\pi}$, $\alpha \in \mathbb{R}$; e) $\lim\limits_{x\to 0+}\left(\frac{\ln(1+x)}{x}\right)^{\frac{1}{x}} = e^{-\frac{1}{2}}$; f) $\lim\limits_{x\to+\infty}\frac{x}{2^x} = 0$;
g) $\lim\limits_{x\to+\infty}\frac{\ln x}{x^\varepsilon} = 0$ for all $\varepsilon > 0$; h) $\lim\limits_{x\to+0} x^\varepsilon \ln x = 0$ for all $\varepsilon > 0$; i) $\lim\limits_{x\to+0}(\ln(1+x))^x = 1$.

Exercise 13.2. Compute the following limits:
a) $\lim\limits_{x\to 0}\frac{\ln(1+x)-x}{x^2}$; b) $\lim\limits_{x\to 0}\frac{e^x - e^{\sin x}}{x-\sin x}$; c) $\lim\limits_{x\to+\infty}\left(x\left(\frac{\pi}{2} - \arctan x\right)\right)$; d) $\lim\limits_{x\to+\infty}\frac{\ln(x+1)-\ln(x-1)}{\sqrt{x^2+1}-\sqrt{x^2-1}}$;
e) $\lim\limits_{x\to+\infty}\left(x\sin\frac{1}{x} + \frac{1}{x}\right)^x$; f) $\lim\limits_{x\to+\infty}\left(x\sin\frac{1}{x} + \frac{1}{x^2}\right)^x$; g) $\lim\limits_{x\to 0}\frac{(1+x)^{\frac{1}{x}} - e}{x}$; h) $\lim\limits_{x\to+\infty}\frac{x^{\ln x}}{(\ln x)^x}$.

13.2 Higher Order Derivatives

We assume that a function $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b). We denote its derivative f' by g, that is $g(x) = f'(x)$, $x \in (a, b)$.

Definition 13.1. If there exists a derivative $g'(x_0)$ of the function g at a point x_0, then this derivative is called the **second derivative of** f **at the point** x_0 and is denoted by $f''(x_0)$ or $\frac{d^2 f}{dx^2}(x_0)$.

Let the n-th derivative $f^{(n)}$ be defined on (a, b). Then the $(n+1)$**-th derivative of** f **at** $x_0 \in (a, b)$ is defined as $f^{(n+1)}(x_0) = \frac{d(f^{(n)})}{dx}(x_0)$, if it exists.

Example 13.2. Let $a > 0$. Then for each $x \in \mathbb{R}$ we obtain $(a^x)' = a^x \ln a$, $(a^x)'' = a^x \ln^2 a$, $(a^x)''' = a^x \ln^3 a$, ..., $(a^x)^{(n)} = a^x \ln^n a$. In particular, $(e^x)^{(n)} = e^x$, $x \in \mathbb{R}$.

Exercise 13.3. Let $\alpha \in \mathbb{R}$. Show that $(x^\alpha)^n = \alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)x^{\alpha-n}$ for all $x > 0$ and $n \in \mathbb{N}$.

Example 13.3. Let $\alpha \in \mathbb{R}$. Then $((1+x)^\alpha)^{(n)} = \alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)(1+x)^{\alpha-n}$ for all $x > -1$ and $n \in \mathbb{N}$.

Indeed, $((1+x)^\alpha)' = \alpha(1+x)^{\alpha-1}$, $((1+x)^\alpha)'' = \left(\alpha(1+x)^{\alpha-1}\right)' = \alpha(\alpha-1)(1+x)^{\alpha-2}$ and so on.

Exercise 13.4. Show that $(\ln(1+x))^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ for all $x > -1$ and $n \in \mathbb{N}$.

Example 13.4. For each $x \in \mathbb{R}$ $(\sin x)^{(n)} = \sin\left(x + n\frac{\pi}{2}\right)$ and $(\cos x)^{(n)} = \cos\left(x + n\frac{\pi}{2}\right)$.

Indeed, $(\sin x)' = \cos x = \sin\left(x + \frac{\pi}{2}\right)$, $(\sin x)'' = (\cos x)' = -\sin x = \sin\left(x + 2\frac{\pi}{2}\right)$, $(\sin x)''' = (-\sin x)' = -\cos x = \sin\left(x + 3\frac{\pi}{2}\right)$ and so on. The same computation for $(\cos x)^{(n)}$.

Exercise 13.5. Compute the n-th derivative of the following functions:
a) $f(x) = 2^{x-1}$, $x \in \mathbb{R}$; b) $f(x) = \sqrt{1+x}$, $x > -1$; c) $f(x) = \arctan x$, $x \in \mathbb{R}$.

Theorem 13.2. *Let functions $f, g : (a, b) \to \mathbb{R}$ have n-th derivatives on (a, b). Then the following equalities are true.*

1) for all $k \in \{1, \dots, n\}$ $(f^{(n-k)})^{(k)} = (f^{(k)})^{(n-k)} = f^{(n)}$, where $f^{(0)} = f$;

2) for all $c \in \mathbb{R}$ $(cf)^{(n)} = cf^{(n)}$;

3) $(f+g)^{(n)} = f^{(n)} + g^{(n)}$.

Theorem 13.3 (Leibniz Formula)**.** *For a number $n \in \mathbb{N}$ let $g, f : (a,b) \to \mathbb{R}$ have n-th derivatives on (a,b). Then $f \cdot g$ has the n-th derivative on (a,b) and*

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(k)} g^{(n-k)},$$

where $C_n^k = \frac{n!}{k!(n-k)!}$.

Exercise 13.6. Compute the following derivatives:
a) $(x^2 e^x)^{(n)}$, $x \in \mathbb{R}$; b) $(x^3 \sin x)^{(n)}$, $x \in \mathbb{R}$; c) $(x^n \ln x)^{(n)}$, $x > 0$.

13.3 Taylor's Formula

13.3.1 Taylor's Formula for a Polynomial

Let $n \in \mathbb{N}$ and $\{a_0, a_1, a_2, \dots, a_n\} \subset \mathbb{R}$. For any point $x_0 \in \mathbb{R}$ a polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n, \quad x \in \mathbb{R},$$

can be written in the form

$$P(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \dots + b_n(x - x_0)^n, \quad x \in \mathbb{R}, \tag{11}$$

where $\{b_0, b_1, b_2, \dots, b_n\}$ are some real numbers, which can be computed by the following way. Inserting $x = x_0$ into (11), we obtain $b_0 = P(x_0)$. Next we compute P'. So,

$$P'(x) = b_1 + 2b_2(x - x_0) + 3b_3(x - x_0)^2 + \dots + nb_n(x - x_0)^{n-1}, \quad x \in \mathbb{R}. \tag{12}$$

Inserting $x = x_0$ into (12), we get $b_1 = P'(x_0)$. Next, we compute the second derivative of P

$$P''(x) = 2b_2 + 3 \cdot 2 \cdot b_3(x - x_0) + \dots + n(n-1)b_n(x - x_0)^{n-2}, \quad x \in \mathbb{R}. \tag{13}$$

Inserting $x = x_0$ into (13), we obtain $b_2 = \frac{P''(x_0)}{2}$. Similarly, we obtain

$$b_k = \frac{P^{(k)}(x_0)}{k!}, \quad k \geq 0.$$

Thus, for each $x \in \mathbb{R}$

$$P(x) = P(x_0) + \frac{P'(x_0)}{1!}(x - x_0) + \frac{P''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{P^{(n)}(x_0)}{n!}(x - x_0)^n. \tag{14}$$

We see that any polynomial can be completely defined only by its value and values of its derivatives at a point x_0. Formula (14) does not hold if P is not a polynomial, but it turns out that values of a function are close to the right hand side of (14) if x is close to x_0.

13.3.2 Taylor's Formula with Peano Remainder Term

Let $f, g : A \to \mathbb{R}$ be some functions and x_0 be a limit point of A. If $\frac{f(x)}{g(x)} \to 0$, $x \to x_0$, then we will write $f(x) = o(g(x))$, $x \to 0$, or $f = o(g)$, $x \to x_0$.

Exercise 13.7. Show that
a) $x = o(1)$, $x \to 0$; b) $x^3 = o(2^x)$, $x \to +\infty$; c) $\ln x = o(\sqrt{x})$, $x \to +\infty$; d) $x - \sin x = o(x)$, $x \to 0$.

Theorem 13.4. *Let $n \in \mathbb{N}$ and let a function $f : (a, b) \to \mathbb{R}$ and a point $x_0 \in (a, b)$ satisfy the following conditions:*

1) there exists $f^{(n-1)}(x)$ for all $x \in (a, b)$;

2) there exists $f^{(n)}(x_0)$.

Then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + o((x - x_0)^n), \quad x \to x_0. \tag{15}$$

The term $o((x - x_0)^n)$ is called the **Peano remainder term**.

Proof. We recall that $0! = 1$ and set

$$R_n(x) := f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k, \quad x \in (a, b).$$

According to assumptions 1) and 2), there exists $R^{(n-1)}(x)$ for all $x \in (a, b)$ and $R^{(n)}(x_0)$. Moreover it is easy to see that

$$R_n(x_0) = R_n'(x_0) = R_n''(x_0) = \ldots = R_n^{(n)}(x_0) = 0.$$

Assuming $x > x_0$ and applying the Lagrange theorem (see Theorem 11.4), we have

$$\begin{aligned}
\left|\frac{R_n(x)}{(x - x_0)^n}\right| &= \left|\frac{R_n(x) - R_n(x_0)}{(x - x_0)^n}\right| = \left|\frac{R_n'(c_1)(x - x_0)}{(x - x_0)^n}\right| = \left|\frac{R_n'(c_1) - R_n'(x_0)}{(x - x_0)^{n-1}}\right| \\
&= \left|\frac{R_n''(c_2)(c_1 - x_0)}{(x - x_0)^{n-1}}\right| \le \left|\frac{R_n''(c_2)}{(x - x_0)^{n-2}}\right| = \left|\frac{R_n''(c_2) - R_n''(x_0)}{(x - x_0)^{n-2}}\right| = \left|\frac{R_n'''(c_3)(c_2 - x_0)}{(x - x_0)^{n-2}}\right| \le \ldots \\
&\le \left|\frac{R_n^{(n-1)}(c_{n-1}) - R_n^{(n-1)}(x_0)}{x - x_0}\right| \to \left|R_n^{(n)}(x_0)\right| = 0, \quad x \to x_0+,
\end{aligned}$$

where $x_0 < c_{n-1} < c_{n-2} < \ldots < c_2 < c_1 < x$. Moreover $c_{n-1} \to x_0$ as $x \to x_0+$.

One can similarly obtain that $\left|\frac{R_n(x)}{(x-x_0)^n}\right| \to 0$, $x \to x_0-$. Consequently,

$$R_n(x) = o((x - x_0)^n), \quad x \to x_0,$$

by Theorem 7.8. □

Example 13.5. For every $n \in \mathbb{N}$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + o(x^n), \quad x \to 0.$$

The formula follows from Theorem 13.4 applying to $f(x) = e^x$, $x \in \mathbb{R}$, and the fact that $f^{(k)}(0) = e^0 = 1$ (see Example 13.2).

Example 13.6. For all $n \in \mathbb{N}$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + o(x^n), \quad x \to 0.$$

The formula follows from Theorem 13.4 applying to $f(x) = \ln(1+x)$, $x > -1$, and the fact that $f^{(k)}(0) = \frac{(-1)^{k-1}(k-1)!}{(1+0)^k} = (-1)^{k-1}(k-1)!$ (see Example 13.4).

Example 13.7. For each $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)x^n}{n!} + o(x^n), \quad x \to 0.$$

The formula follows from Theorem 13.4 applying to $f(x) = (1+x)^\alpha$, $x > -1$, and the fact that $f^{(k)}(0) = \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)(1+0)^{\alpha-k} = \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)$ (see Example 13.3).

Exercise 13.8. Show that for every $n \in \mathbb{N} \cup \{0\}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}), \quad x \to 0,$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1}), \quad x \to 0.$$

Exercise 13.9. Show that for every $n \in \mathbb{N} \cup \{0\}$

$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2}), \quad x \to 0,$$
$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + o(x^{2n+1}), \quad x \to 0.$$

Exercise 13.10. Use Taylor's formula to compute the limits:
a) $\lim\limits_{x\to 0} \frac{e^x-1-x}{x^2}$; b) $\lim\limits_{x\to 0} \frac{x-\sin x}{e^x-1-x-\frac{x^2}{2}}$; c) $\lim\limits_{x\to 0} \frac{\ln(1+x+x^2)-\ln(1-x-x^2)}{x\sin x}$; d) $\lim\limits_{x\to 0} \frac{\cos(xe^x)-\cos(xe^{-x})}{x^3}$.