# 12 Lecture 12 – Application of Derivatives

## 12.1 Applications of Lagrange Theorem

**Corollary 12.1.** *Let a function $f : (a, b) \to \mathbb{R}$ have the derivative $f'$ on $(a, b)$ and for each $x \in (a, b)$ $f'(x) = 0$. Then there exists $L \in \mathbb{R}$ such that $f(x) = L$ for all $x \in (a, b)$.*

*Proof.* Let $x_0 \in (a, b)$ be an arbitrary fixed point and $x \neq x_0$. Applying the Lagrange theorem to the interval with the ends $x_0$ and $x$, we obtain

$$f(x) - f(x_0) = f'(c)(x - x_0) = 0.$$

Thus, we can set $L := f(x_0)$. □

**Corollary 12.2.** *Let functions $f, g : (a, b) \to \mathbb{R}$ have the derivatives $f'$, $g'$ on $(a, b)$ and for each $x \in (a, b)$ $f'(x) = g'(x)$. Then there exists $L \in \mathbb{R}$ such that $f(x) = g(x) + L$ for all $x \in (a, b)$.*

*Proof.* Applying Corollary 12.1 to the function $f - g$, we obtain that there exists a constant $L$ such that $f(x) - g(x) = L$, $x \in (a, b)$. □

**Corollary 12.3.** *Let a function $f : (a, b) \to \mathbb{R}$ have the derivative $f'$ on $(a, b)$ and for each $x \in (a, b)$ $f'(x) = M$, where $M$ is some real number. Then there exists $L \in \mathbb{R}$ such that $f(x) = Mx + L$ for all $x \in (a, b)$.*

*Proof.* Applying Corollary 12.2 to the functions $f$ and $g(x) = Mx$, $x \in (a, b)$, we obtain the statement. □

**Exercise 12.1.** Let $a, b$ be a fixed numbers. Identify all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = ax + b$, $x \in \mathbb{R}$.

**Exercise 12.2.** Identify all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(x)$, $x \in \mathbb{R}$.
(*Hint:* Note that $(f(x)e^{-x})' = (f'(x) - f(x))e^{-x}$, $x \in \mathbb{R}$)

**Exercise 12.3.** Let functions $f, g : (a, b) \to (0, +\infty)$ be differentiable on $(a, b)$ and for every $x \in (a, b)$ $\frac{f'(x)}{f(x)} = \frac{g'(x)}{g(x)}$. Prove that there exists $L > 0$ such that $f(x) = Lg(x)$ for all $x \in (a, b)$.
(*Hint:* Consider the functions $\ln f$ and $\ln g$)

## 12.2 Proofs of Inequalities

In this section, we are going to prove a couple of inequalities which are often used in mathematics.

**Example 12.1.** We prove that for all $x_1, x_2 \in \mathbb{R}$
a) $|\sin x_1 - \sin x_2| \leq |x_1 - x_2|$; b) $|\cos x_1 - \cos x_2| \leq |x_1 - x_2|$; c) $|\arctan x_1 - \arctan x_2| \leq |x_1 - x_2|$.

The proof of these inequalities are similar. So, we will prove only a). We assume that $x_1 < x_2$. Then applying the Lagrange theorem to the function $f(x) = \sin x$, $x \in [x_1, x_2]$, we have that there exists $c \in (x_1, x_2)$ such that

$$|\sin x_2 - \sin x_1| = |\cos c| \cdot |x_2 - x_1| \leq |x_2 - x_1|,$$

since $|\cos c| \leq 1$.

**Exercise 12.4.** Prove b) and c) in Example 12.1.

**Exercise 12.5.** Prove that
a) $|\sqrt{x_1} - \sqrt{x_2}| \leq \frac{1}{2}|x_1 - x_2|$ for all $x_1, x_2 \in [1, +\infty)$;
b) $|\sqrt{u^2+v^2} - \sqrt{u^2+w^2}| \leq |v - w|$ for all $u, v, w \in \mathbb{R}$. (*Hint:* Consider the function $f(t) = \sqrt{u^2+t^2}$, $t \in \mathbb{R}$)

**Example 12.2.** We prove that
a) $e^x \geq 1 + x$ for all $x \in \mathbb{R}$, where $e^x = 1 + x$ only if $x = 0$; b) $e^x > 1 + x + \frac{x^2}{2}$ for all $x > 0$.

We prove a). We first assume that $x > 0$. Then applying the Lagrange theorem to the function $f(u) = e^u$, $u \in [0, x]$, we obtain that there exists $c \in (0, x)$ such that $e^x - e^0 = e^c \cdot (x - 0)$. Since $e^c > 1$ for $c > 0$, we obtain $e^x - 1 > x$ for all $x > 0$. Next let $x < 0$. Then we can apply the Lagrange theorem to the function $f(u) = e^u$, $u \in [x, 0]$. So, we obtain that there exists $c \in (x, 0)$ such that $e^0 - e^x = e^c \cdot (0 - x)$. Since $e^c < 1$ for $c < 0$, we get $1 - e^x < -x$.

In order to prove b), we apply the Cauchy theorem to the functions $f(u) = e^u$, $g(u) = 1 + u + \frac{u^2}{2}$, $u \in [0, x]$. Hence, there exists $c \in (0, x)$ such that

$$\frac{e^x - e^0}{1 + x + \frac{x^2}{2} - 1} = \frac{e^c}{1 + c},$$

Using a), we have $e^x - 1 > x + \frac{x^2}{2}$.

**Exercise 12.6.** Prove that $e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$ for all $x > 0$ and $n \in \mathbb{N}$.
(*Hint:* Use Example 12.2 and mathematical induction)

**Exercise 12.7.** Prove that $\frac{x}{1+x} \leq \ln(1 + x) \leq x$ for all $x > -1$.

**Exercise 12.8** (Generalised Bernoulli inequality)**.** For each $\alpha > 1$, prove that $(1 + x)^\alpha \geq 1 + \alpha x$ for all $x > -1$. Moreover, $(1 + x)^\alpha = 1 + \alpha x$ iff $x = 0$.

**Exercise 12.9.** Prove that
a) $x - \frac{x^3}{3!} \leq \sin x \leq x$ for all $x \geq 0$;
b) $1 - \frac{x^2}{2} \leq \cos x \leq 1$ for all $x \geq 0$.

## 12.3 Investigation of Monotonicity of Functions

**Theorem 12.1.** *Let $-\infty \leq a < b \leq +\infty$ and a function $f : (a, b) \to \mathbb{R}$ be differentiable on $(a, b)$.*

*(i) The function $f$ increases on $(a, b)$ iff $f'(x) \geq 0$ for all $x \in (a, b)$.*

*(ii) The function $f$ decreases on $(a, b)$ iff $f'(x) \leq 0$ for all $x \in (a, b)$.*

*Proof.* We prove (i). Let first $f'(x) \geq 0$ for all $x \in (a, b)$. We take $x_1, x_2 \in (a, b)$ and $x_1 < x_2$. Then applying the Lagrange theorem to the function $f$ on the interval $[x_1, x_2]$, we have that there exists $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \geq 0. \tag{10}$$

Next, let $f$ increases on $(a, b)$. Then for each $x_0 \in (a, b)$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

Here we used the definition of derivative, Remark 10.2 and the fact that $f(x) \geq f(x_0)$ for $x > x_0$.

In order to prove (ii), apply (i) of the theorem to the function $g(x) = -f(x)$, $x \in (a, b)$. □

**Remark 12.1.** a) If $f'(x) > 0$ for all $x \in (a, b)$, then the function $f$ is strictly increasing.

b) If $f'(x) < 0$ for all $x \in (a, b)$, then the function $f$ is strictly decreasing.
Indeed, a) immediately follows from (10), where we have the strict inequality.

We note that the inverse statements of Remark 12.1 is not valid. Indeed, the function $f(x) = x^3$, $x \in \mathbb{R}$, strictly increases but its derivative $f'(x) = 3x^2$, $x \in \mathbb{R}$, equals 0 at $x = 0$.

We formulate more general statement about strictly monotone functions.

**Theorem 12.2.** *Let $-\infty \leq a < b \leq +\infty$ and a function $f : (a, b) \to \mathbb{R}$ be differentiable on $(a, b)$.*

*(i) The function $f$ strictly increases on $(a, b)$ iff $f'(x) \geq 0$ for all $x \in (a, b)$ and there exists no interval $(\alpha, \beta) \subset (a, b)$ such that $f'(x) = 0$ for all $x \in (\alpha, \beta)$.*

*(ii) The function $f$ strictly decreases on $(a, b)$ iff $f'(x) \leq 0$ for all $x \in (a, b)$ and there exists no interval $(\alpha, \beta) \subset (a, b)$ such that $f'(x) = 0$ for all $x \in (\alpha, \beta)$.*

**Example 12.3.** By Theorem 12.2, the function $f(x) = x^2 + bx + c$, $x \in \mathbb{R}$, strictly decreases on $\left(-\infty, -\frac{b}{2}\right]$ and strictly increases on $\left[-\frac{b}{2}, +\infty\right)$, since $f'(x) = 2x + b < 0$ for $x < -\frac{b}{2}$ and $f'(x) = 2x + b > 0$ for $x > -\frac{b}{2}$

**Example 12.4.** By Theorem 12.2, the function $f(x) = e^x$, $x \in \mathbb{R}$, is strictly increasing on $\mathbb{R}$, since $f'(x) = e^x > 0$, $x \in \mathbb{R}$.

**Example 12.5.** By Theorem 12.2, the function $f(x) = x + \sin x$, $x \in \mathbb{R}$, is strictly increasing on $\mathbb{R}$, since $f'(x) = 1 + \cos x > 0$ for all $x \in \mathbb{R} \setminus \{x : \cos x = -1\} = \mathbb{R} \setminus \{(2k+1)\pi : k \in \mathbb{Z}\}$.

**Example 12.6.** The function $f(x) = \frac{\ln x}{x}$, $x > 0$, strictly increases on $(0, e]$ and strictly decreases on $[e, +\infty)$ according to Theorem 12.2. Indeed, its derivative $f'(x) = \frac{1-\ln x}{x^2}$, $x > 0$, is strictly positive on $(0, e)$ and strictly negative on $(e, +\infty)$.

**Example 12.7.** The function $f(x) = x^x$, $x > 0$, is strictly increasing on $\left[\frac{1}{e}, +\infty\right)$ and strictly decreasing on $\left(-\infty, \frac{1}{e}\right]$ according to Theorem 12.2. Indeed, its derivative $f'(x) = x^x(1 + \ln x)$, $x > 0$, is strictly positive on $\left(\frac{1}{e}, +\infty\right)$ and strictly negative on $\left(-\infty, \frac{1}{e}\right)$. For the computation of the derivative see Example 11.4.

**Exercise 12.10.** Identify intervals on which the following functions are monotone.
a) $f(x) = x^2 - x$, $x \in \mathbb{R}$; b) $f(x) = \frac{x}{1+x^2}$, $x \in \mathbb{R}$; c) $f(x) = \frac{1}{x^3} - \frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$;
d) $f(x) = x + \sqrt{|1 - x^2|}$, $x \in \mathbb{R}$.

**Exercise 12.11.** Identify $a \in \mathbb{R}$ for which the function $f(x) = x + a \sin x$, $x \in \mathbb{R}$, is increasing on $\mathbb{R}$.