11 Lecture 11 – Derivatives of Inverse Functions and some Theorems

11.1 Derivative of Inverse Function

Theorem 11.1 (Differentiation of inverse function). *Let* $-\infty \leq a < b \leq +\infty$ *and a function* $f : (a, b) \to \mathbb{R}$ *satisfy the following properties*

1) f *is continuous on* (a, b)*;*

2) f *strictly increases on* (a, b)*.*

Let $(c, d) := f((a, b)) = \{f(x) : \ x \in (a, b)\}$*, where* $-\infty \leq c < d \leq +\infty$*. Let also* $g : (c, d) \to (a, b)$ *be the inverse function to* f*.*

If there exists a derivative $f'(x_0) \neq 0$ *at a point* $x_0 \in (a, b)$*, then the function* g *has a derivative* $g'(y_0)$ *at the point* $y_0 = f(x_0)$*. Moreover,*

$$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(g(y_0))}.$$

Remark 11.1. If a function $f : (a, b) \to \mathbb{R}$ is continuous and strictly increasing, then, by Theorem 8.5, the range $f((a, b))$ of f is an interval and there exists the inverse function g to f which is also continuous and strictly increasing.

Proof of Theorem 11.1. Since the function g is strictly increasing (see Remark 11.1), we have that $g(y) \neq g(y_0)$ for $y \neq y_0$. Using the definition of inverse function and Theorem 10.1, we obtain

$$y - y_0 = f(g(y)) - f(g(y_0)) = f'(g(y_0))(g(y) - g(y_0)) + \varphi(g(y))(g(y) - g(y_0)),$$

where $\varphi(g(y)) \to 0$ as $g(y) \to g(y_0)$. Since g is continuous on (c, d), one has $g(y) \to g(y_0)$, $y \to y_0$. Thus, $\varphi(g(y)) \to 0$, $y \to y_0$. Consequently,

$$\begin{aligned}\frac{g(y) - g(y_0)}{y - y_0} &= \frac{g(y) - g(y_0)}{f'(g(y_0))(g(y) - g(y_0)) + \varphi(g(y))(g(y) - g(y_0))} \\ &= \frac{1}{f'(g(y_0)) + \varphi(g(y))} \to \frac{1}{f'(g(y_0))}, \quad y \to y_0.\end{aligned}$$

□

Remark 11.2. Let us assume that, in Theorem 11.1, the function f has a derivative $f'(x) \neq 0$ at each point $x \in (a, b)$. Then for each $y \in (c, d)$ there exists the derivative $g'(y)$ and one can get a relationship between f' and g' using the equalities $g(f(x)) = x$, $x \in (a, b)$, and $f(g(y)) = y$, $y \in (c, d)$. Indeed, by the chain rule (see Theorem 10.4), $g'(f(x))f'(x) = 1$, $x \in (a, b)$, and $f'(g(y))g'(y) = 1$, $y \in (c, d)$.

Example 11.1. Let $\alpha > 0$, $\alpha \neq 1$. Then $(\log_\alpha x)' = \frac{1}{x \ln \alpha}$ for all $x > 0$. In particular, $(\ln x)' = \frac{1}{x}$ for all $x > 0$.

We will consider the case $\alpha > 1$. To compute the derivative $(\log_\alpha x)'$, we are going to use Theorem 11.1. So, we set $f(x) := \alpha^x$, $x \in (a, b) := \mathbb{R}$ and $(c, d) := (0, +\infty)$. Then f is continuous and strictly increasing on $\mathbb{R}$. Moreover, $f'(x) = \alpha^x \ln \alpha \neq 0$, $x \in \mathbb{R}$, by Example 10.7. So, applying

Theorem 11.1, to the function $g(y) = \log_\alpha y$, $y \in (c, d) = (0, +\infty)$, which is inverse to f, we get for $y_0 \in (0, +\infty)$

$$g'(y_0) = (\log_\alpha y_0)' = \frac{1}{f'(x_0)} = \frac{1}{\alpha^{x_0} \ln \alpha} = \frac{1}{y_0 \ln \alpha},$$

where $y_0 = f(x_0) = \alpha^{x_0}$.

Exercise 11.1. Show that $(\log_\alpha x)' = \frac{1}{x \ln \alpha}$, $x > 0$, for $0 < \alpha < 1$.

Example 11.2. For all $x \in \mathbb{R}$ $(\arctan x)' = \frac{1}{1+x^2}$.

Again we are going to use Theorem 11.1. We set $f(x) := \tan x$, $x \in (a, b) := \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $(c, d) := \mathbb{R}$. By Example 8.5, f is continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Moreover, it is strictly increasing and $f'(x) = \frac{1}{\cos^2 x} \neq 0$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Thus, applying Theorem 11.1 to $g(y) = \arctan y$, $y \in \mathbb{R}$, we have for each $y_0 \in \mathbb{R}$

$$g'(y_0) = (\arctan y_0)' = \frac{1}{f'(x_0)} = \cos^2 x_0 = \frac{1}{1 + \tan^2 x_0} = \frac{1}{1 + y_0^2},$$

where $y_0 = f(x_0) = \tan x_0$.

Exercise 11.2. For all $x \in \mathbb{R}$ $(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$.

Example 11.3. For each $x \in (-1, 1)$ $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$.

We set $f(x) := \sin x$, $x \in (a, b) := \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $(c, d) := (-1, 1)$. By Example 8.5, f is continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Moreover, f is strictly increasing and $f'(x) = \cos x \neq 0$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, by Example 10.8. Thus, applying Theorem 11.1 to the function $g(x) = \arcsin y$, $y \in (-1, 1)$, we obtain for $y_0 \in (-1, 1)$

$$g'(y_0) = (\arcsin y_0)' = \frac{1}{f'(x_0)} = \frac{1}{\cos x_0} = \frac{1}{\sqrt{1 - \sin^2 x_0}} = \frac{1}{\sqrt{1 - y_0^2}},$$

where $y_0 = f(x_0) = \sin x_0$.

Exercise 11.3. Show that for each $x \in (-1, 1)$ $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$.

Example 11.4. Compute the derivative of the function $f(x) = x^x$, $x > 0$.

Solution. For $x > 0$ we have $(x^x)' = \left(e^{\ln x^x}\right)' = \left(e^{x \ln x}\right)' = e^{x \ln x}(x \ln x)' = x^x \left((x)' \ln x + x(\ln x)'\right) = x^x \left(\ln x + x\frac{1}{x}\right) = x^x (\ln x + 1)$.

11.2 Some Theorems

Theorem 11.2 (Fermat theorem)**.** *Let $f : (a, b) \to \mathbb{R}$, $x_0 \in (a, b)$ and $f(x_0) = \max_{x \in (a,b)} f(x)$ or $f(x_0) = \min_{x \in (a,b)} f(x)$. If f has a derivative at the point x_0, then $f'(x_0) = 0$.*

Proof. We assume that $f(x_0) = \max_{x \in (a,b)} f(x)$. Then for each $x \in (a, b)$ $f(x) \leq f(x_0)$. Thus, by Remark 10.2, we have

$$f'(x_0) = f'_-(x_0) = \lim_{x \to x_0-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

and, similarly,

$$f'(x_0) = f'_+(x_0) = \lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0.$$

This implies that $f'(x_0) = 0$.

The case $f(x_0) = \min_{x \in (a,b)} f(x)$ is similar. □

Remark 11.3. In the Fermat theorem, the assumption $a < x_0 < b$ is essential. Indeed, the statement is not valid for the function $f(x) = x$, $x \in [0, 1]$. In that case, $x_0 = 1$, but $f'(x_0) = 1$.

Theorem 11.3 (Rolle's theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ satisfies the following properties*

1) f is continuous on $[a, b]$;

2) for each $x \in (a, b)$ the derivative $f'(x)$ exists;

3) $f(a) = f(b)$.

Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof. If for every $x \in [a, b]$ $f(x) = f(a)$, then f is a constant function. Consequently, $f'(c) = 0$ for all $c \in (a, b)$.

We now assume that

$$\exists x \in [a, b] \quad \text{such that} \quad f(x) \neq f(a). \tag{9}$$

According to the assumption 1) and the 2nd Weierstrass theorem (see Theorem 9.2), there exist $x_*, x^* \in [a, b]$ such that $f(x_*) = \min_{x\in[a,b]} f(x)$ and $f(x^*) = \max_{x\in[a,b]} f(x)$. Using assumptions (9) and 3), we have that $f(x_*) \neq f(a)$ or $f(x^*) \neq f(a)$. We consider the case $f(x^*) \neq f(a)$. In this case, we have $x^* \neq a$ and $x^* \neq b$, which implies that $x^* \in (a, b)$. Hence, the function f and the point $x_0 = x^*$ satisfy all assumptions of the Fermat theorem (see Theorem 11.2). Consequently, $f'(x^*) = 0$. We take $c := x^*$.

The case $f(x_*) \neq f(a)$ can be considered similarly. □

Exercise 11.4. Let a function $f \in C([a, b])$ have the derivative $f'(x) \neq 0$ for all $x \in (a, b)$. Then $f(a) \neq f(b)$.

Theorem 11.4 (Lagrange (mean value) theorem)**.** *We assume that a function $f : [a, b] \to \mathbb{R}$ satisfies the following properties*

1) f is continuous on $[a, b]$;

2) f is differentiable on (a, b), that is, f has a derivative $f'(x)$ for all $x \in (a, b)$.

Then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

Proof. We take

$$g(x) := f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a), \quad x \in [a, b],$$

and note that g satisfies the assumptions of Rolle's theorem (see Theorem 11.3). Moreover,

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}, \quad x \in (a, b).$$

By Rolle's theorem, there exists $c \in (a, b)$ such that $g'(c) = 0$. It implies that $f'(c) - \frac{f(b)-f(a)}{b-a} = 0$. Consequently, $f(b) - f(a) = f'(c)(b - a)$. □

Exercise 11.5. Let a function $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b) and there exists $L \in \mathbb{R}$ such that $|f'(x)| \le L$ for all $x \in (a, b)$. Show that f is uniformly continuous on (a, b).

Theorem 11.5 (Cauchy theorem)**.** *Let functions* $f, g : [a, b] \to \mathbb{R}$ *satisfy the following conditions*

1) f, g *are continuous on* $[a, b]$*;*

2) f, g *are differentiable on* (a, b)*;*

3) for every $x \in (a, b)$ $g'(x) \neq 0$*.*

Then there exists $c \in (a, b)$ *such that* $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$*.*

Proof. We first note that $g(a) \neq g(b)$. Otherwise, if $g(a) = g(b)$, then there exists $c \in (a, b)$ such that $g'(c) = 0$, by Rolle's theorem. But this contradicts assumption 3).

So, we can set

$$h(x) := f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)), \quad x \in [a, b].$$

Then the function h satisfies the assumptions of Rolle's theorem. Consequently, there exists $c \in (a, b)$ such that $h'(c) = 0$. Thus, $f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) = 0$. □