10 Lecture 10 – Differentiation

10.1 Definition and Some Examples

Let $A \subset \mathbb{R}$ and $a \in A$. We also assume that there exists $\delta > 0$ such that $(a - \delta, a + \delta) \subset A$. Let $f : A \to \mathbb{R}$ be a given function.

Definition 10.1.
- We say that f is **differentiable at** a, or f has a **derivative at** a, if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

  exists and is finite. We will write $f'(a)$ or $\frac{df}{dx}(a)$ for the derivative of f at a, that is,

$$f'(a) := \frac{df}{dx}(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

  whenever this limit exists and is finite.

- If for each $a \in A$ the derivative $f'(a)$ exists, then the function f is said to be **differentiable on** A and the function defined by $A \ni x \mapsto f'(x)$ is called the **derivative of** f **on the set** A.

Remark 10.1. Taking $h := x - a$ in Definition 10.1, we have

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

Definition 10.2.
- If a finite left-sided limit

$$\lim_{x \to a-} \frac{f(x) - f(a)}{x - a}$$

  exists, then this limit is called a **left derivative of** f **at** a and is denoted by $f'_-(a)$ or $\frac{d^- f}{dx}(a)$.

- If a finite right-sided limit

$$\lim_{x \to a+} \frac{f(x) - f(a)}{x - a}$$

  exists, then this limit is called a **right derivative of** f **at** a and is denoted by $f'_+(a)$ or $\frac{d^+ f}{dx}(a)$.

Remark 10.2. By Theorem 7.8, a derivative $f'(a)$ exists iff $f'_-(a)$ and $f'_+(a)$ exist and $f'_-(a) = f'_+(a)$.

Example 10.1. For the function $f(x) = x$, $x \in \mathbb{R}$, and any point $a \in \mathbb{R}$ we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x - a}{x - a} = 1.$$

Thus, $(x)' = 1$, $x \in \mathbb{R}$.

Example 10.2. For the function $f(x) = |x|$, $x \in \mathbb{R}$, and the point $a = 0$ we have

$$f'_-(0) = \lim_{x \to 0-} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0-} \frac{-x}{x} = -1$$

and

$$f'_+(0) = \lim_{x \to 0+} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0+} \frac{x}{x} = 1.$$

So, $f'_-(0) = -1 \neq f'_+(0) = 1$ and the derivative $f'(0)$ does not exist.

Example 10.3. Let $f(x) = x^2$, $x \in \mathbb{R}$, and $a \in \mathbb{R}$. Then

$$f'(a) = \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} \frac{(x-a)(x+a)}{x - a} = \lim_{x \to a}(x + a) = 2a.$$

Hence, $(x^2)' = 2x$, $x \in \mathbb{R}$.

Example 10.4. For the function $f(x) = \sqrt[3]{x}$, $x \in \mathbb{R}$, and the point $a = 0$ we have

$$\lim_{x \to 0} \frac{\sqrt[3]{x} - \sqrt[3]{0}}{x - 0} = \lim_{x \to 0} \frac{1}{\sqrt[3]{x^2}} = +\infty.$$

Consequently, the derivative of f at 0 does not exist.

Example 10.5. Let $f(x) = x \sin \frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$, and $f(0) = 0$. Let also $a = 0$. Then $\frac{f(x) - f(0)}{x - 0} = \frac{x \sin \frac{1}{x} - 0}{x - 0} = \sin \frac{1}{x}$ does not have any limit as $x \to 0$ (see Exercise 8.8). Thus, the function f is not differentiable at 0.

Exercise 10.1. Check that $(x|x|)' = 2|x|$, $x \in \mathbb{R}$.

Exercise 10.2. For the function $f(x) = |x^2 - x|$, $x \in \mathbb{R}$, compute $f'(x)$ for each $x \in \mathbb{R} \setminus \{0, 1\}$. Compute left and right derivatives at points 0 and 1.

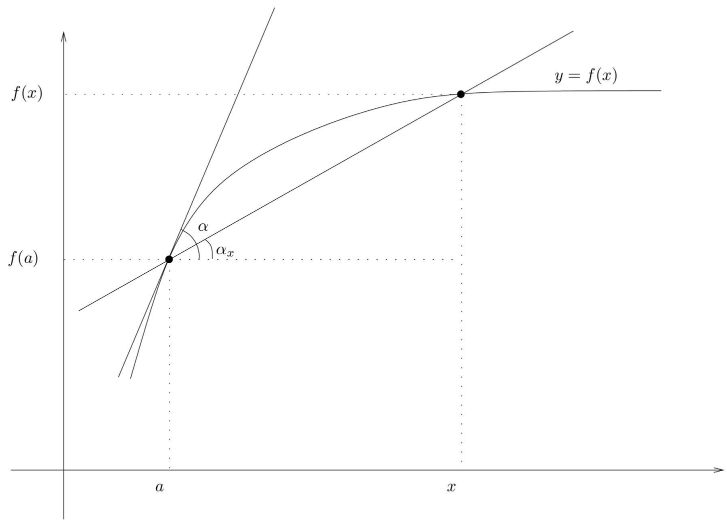
10.2 Interpretation of Derivative

a) **Physical Interpretation**.

Let a point P move on the real line and $s(t)$ is its position at time t. Let t_1, t_2 be two moments of time and $t_1 < t_2$. Then the average velocity over the period of time $[t_1, t_2]$ is the ratio $\frac{s(t_2) - s(t_1)}{t_2 - t_1}$, where $s(t_2) - s(t_1)$ is the distance travelled by P during the time $t_2 - t_1$. The instantaneous velocity at t_1 is the limit of the average velocity as t_2 approaches t_1, that is, it is the limit $\lim_{t_2 \to t_1} \frac{s(t_2) - s(t_1)}{t_2 - t_1}$. Thus, **instantaneous velocity** $v(t)$ **of the point** P **at time** t **is the derivative of** s **at** t, i.e. $v(t) = s'(t)$.

b) **Geometric interpretation**

Let a function $f : (a, b) \to \mathbb{R}$ be differentiable at a.



The slope of the secant line through $(a, f(a))$ and $(x, f(x))$ is

$$\tan \alpha_x = \frac{f(x) - f(a)}{x - a}.$$

If x approaches a, the secant line through $(a, f(a))$ and $(x, f(x))$ approaches the tangent line through $(a, f(a))$. Hence, the **derivative** $f'(a)$ **is the slope of the tangent line through the point** $(a, f(a))$, that is,

$$\tan \alpha = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a).$$

So, the linear function whose graph is the tangent line through $(a, f(a))$ can be given by the equation

$$f'(a) = \tan \alpha = \frac{y - f(a)}{x - a}, \quad x \in \mathbb{R},$$

that is,

$$y = f(a) + f'(a)(x - a), \quad x \in \mathbb{R}.$$

10.3 Properties of Derivatives

Theorem 10.1. *If a function $f : A \to \mathbb{R}$ is differentiable at a, then there exists a function $\varphi : A \to \mathbb{R}$ such that*

$$f(x) = f(a) + f'(a)(x - a) + \varphi(x)(x - a), \quad x \in A$$

and $\varphi(x) \to 0$, $x \to a$.

Proof. We take $\varphi(x) := \frac{f(x)-f(a)}{x-a} - f'(a)$, $x \in A \setminus \{a\}$ and $\varphi(a) = 0$. Then the statement easily follows from Definition 10.1. □

Exercise 10.3. If there exists $L \in \mathbb{R}$ such that $f(x) = f(a) + L(x - a) + \varphi(x)(x - a)$, $x \in A$, for some function $\varphi : A \to \mathbb{R}$ satisfying $\varphi(x) \to 0$, $x \to a$, then f is differentiable at a and $f'(a) = L$. Prove this statement.

Theorem 10.2. *If f is differentiable at a point a, then f is continuous at a.*

Proof. Using theorems 10.1 and 8.1, we obtain

$$\lim_{x \to a} f(x) = \lim_{x \to a} \big(f(a) + f'(a)(x - a) + \varphi(x)(x - a)\big) = f(a).$$

□

Exercise 10.4. Let f has a derivative $f'(a)$ at a point a. Express through $f(a)$ and $f'(a)$ the following limits:
a) $\lim\limits_{h \to 0} \frac{f(a-h)-f(a)}{h}$; b) $\lim\limits_{h \to 0} \frac{f(a+2h)-f(a)}{h}$; c) $\lim\limits_{h \to 0} \frac{f(a+h)-f(a-h)}{h}$; d) $\lim\limits_{n \to \infty} n \left(f\left(a + \frac{1}{n}\right) - f(a)\right)$;
e) $\lim\limits_{n \to \infty} n \left(f\left(\frac{n+1}{n} a\right) - f(a)\right)$; f) $\lim\limits_{x \to 1} \frac{f(xa)-f(a)}{x-1}$; g) $\lim\limits_{n \to \infty} \left(\frac{f\left(a+\frac{1}{n}\right)}{f(a)}\right)^n$, $f(a) \neq 0$.

Exercise 10.5. Prove that f is continuous at a point a if $f'_-(a)$ and $f'_+(a)$ exist.

Theorem 10.3 (Differentiation rules). *Let functions $f, g : A \to \mathbb{R}$ have derivatives $f'(a)$ and $g'(a)$ at a point a. Then*

1) *for each $c \in \mathbb{R}$ the function cf has a derivative at a and $(cf)'(a) = cf'(a)$;*
2) *the function $f + g$ has a derivative at a and $(f + g)'(a) = f'(a) + g'(a)$;*
3) *the function $f \cdot g$ has a derivative at a and $(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$;*
4) *if additionally $g(a) \neq 0$, then the function $\frac{f}{g}$ has a derivative at a and $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a)-f(a)g'(a)}{g^2}$.*

Proof. Proof of 2). By the definition of derivative and Theorem 8.1 b), we have

$$(f+g)'(a) \overset{\text{Def. } 10.1}{=} \lim_{x\to a} \frac{f(x)+g(x)-(f(a)+g(a))}{x-a} = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a}\right)$$
$$\overset{\text{Th. } 8.1}{=} \lim_{x\to a}\frac{f(x)-f(a)}{x-a} + \lim_{x\to a}\frac{g(x)-g(a)}{x-a} \overset{\text{Def. } 10.1}{=} f'(a)+g'(a).$$

Proof of 3). We compute

$$(f\cdot g)'(a) \overset{\text{Def. } 10.1}{=} \lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a} = \lim_{x\to a}\frac{f(x)g(x)-f(a)g(x)+f(a)g(x)-f(a)g(a)}{x-a}$$
$$= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}g(x) + f(a)\frac{g(x)-g(a)}{x-a}\right) \overset{\text{Th. } 8.1}{=} \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\lim_{x\to a}g(x)$$
$$+ f(a)\lim_{x\to a}\frac{g(x)-g(a)}{x-a} \overset{\text{(Def. 10.1+Th. 10.2)}}{=} f'(a)g(a)+f(a)g'(a).$$

Proof of 4). Since $g(a) \neq 0$, we have that $g(x) \neq 0$ in some neighbourhood of the point a, by theorems 10.2 and 8.2. Thus,

$$\frac{1}{x-a}\left(\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}\right) = \frac{f(x)g(a)-f(a)g(a)+f(a)g(a)-f(a)g(x)}{g(x)g(a)(x-a)}$$
$$= \frac{1}{g(x)g(a)}\left(\frac{f(x)-f(a)}{x-a}g(a) - f(a)\frac{g(x)-g(a)}{x-a}\right).$$

Thus, the needed equality follows from the latter relation and theorems 10.2 and 8.1. □

Theorem 10.4 (Chain rule). *Let a function $f : A \to \mathbb{R}$ have a derivative $f'(a)$ at a point $a \in A$. Let $f(A) \subset B$ and a function $g : B \to \mathbb{R}$ have a derivative $g'(b)$ at the point $b = f(a)$. Then the composition $g \circ f = g(f)$ has a derivative at the point a and*

$$(g\circ f)'(a) = (g(f))'(a) = g'(f(a))f'(a).$$

Proof. By Theorem 10.1,

$$g(y) - g(b) = g'(b)(y-b) + \varphi(y)(y-b),$$

for some function $\varphi : B \to \mathbb{R}$ satisfying $\varphi(y) \to 0$, $y \to b$. Taking $y := f(x)$ and dividing the latter equality by $x - a$, we obtain

$$\frac{g(f(x))-g(f(a))}{x-a} = g'(b)\frac{f(x)-f(a)}{x-a} + \varphi(f(x))\frac{f(x)-f(a)}{x-a}$$

By Theorem 10.2, $f(x) \to f(a) = b$, $x \to a$, and, consequently, $\varphi(f(x)) \to 0$, $x \to a$. So,

$$(g \circ f)'(a) = \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} = g'(b)f'(a) + 0f'(a) = g'(f(a))f'(a).$$

□

Example 10.6. Let $\alpha \in \mathbb{R}$. Then $(x^\alpha)' = \alpha x^{\alpha-1}$, $x > 0$.

Indeed, using Remark 10.1 and Theorem 8.8, we obtain for $x > 0$

$$\frac{(x+h)^\alpha - x^\alpha}{h} = x^{\alpha-1}\frac{\left(1+\frac{h}{x}\right)^\alpha - 1}{\frac{h}{x}} \to x^{\alpha-1}\cdot\alpha, \quad h \to 0.$$

Exercise 10.6. a) Let $n \in \mathbb{N}$. Show that $(x^n)' = nx^{n-1}$ for all $x \in \mathbb{R}$.
b) Let $m \in \mathbb{Z}$. Show that $(x^m)' = mx^{m-1}$ for all $x \in \mathbb{R} \setminus \{0\}$.

Example 10.7. Let $a > 0$. Then $(a^x)' = a^x \ln a$ for all $x \in \mathbb{R}$. In particular, if $a = e$, then $(e^x)' = e^x$ for all $x \in \mathbb{R}$.

Indeed, using Remark 10.1 and Theorem 8.7, we have for $x \in \mathbb{R}$

$$\frac{a^{x+h} - a^x}{h} = a^x\frac{a^h - 1}{h} \to a^x \cdot \ln a, \quad h \to 0.$$

Example 10.8. a) $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$ for all $x \in \mathbb{R}$;
b) $(\tan x)' = \frac{1}{\cos^2 x}$ for all $x \in \mathbb{R} \setminus \left\{\frac{\pi}{2} + \pi k : \ k \in \mathbb{Z}\right\}$;
c) $(\cot x)' = -\frac{1}{\sin^2 x}$ for all $x \in \mathbb{R} \setminus \{\pi k : \ k \in \mathbb{Z}\}$.

Let us check the equalities in a). For every $x \in \mathbb{R}$ we have

$$\frac{\sin(x+h) - \sin x}{h} = \frac{2}{h}\sin\frac{h}{2}\cos\left(x + \frac{h}{2}\right) = \frac{\sin\frac{h}{2}}{\frac{h}{2}}\cos\left(x + \frac{h}{2}\right) \to \cos x, \quad h \to 0.$$

Thus, $(\sin x)' = \cos x$, $x \in \mathbb{R}$.

Similarly,

$$\frac{\cos(x+h) - \cos x}{h} = -\frac{2}{h}\sin\frac{h}{2}\sin\left(x + \frac{h}{2}\right) = -\frac{\sin\frac{h}{2}}{\frac{h}{2}}\sin\left(x + \frac{h}{2}\right) \to -\sin x, \quad h \to 0.$$

Hence, $(\cos x)' = -\sin x$, $x \in \mathbb{R}$.

In order to compute $(\tan x)'$, we will use Theorem 10.3 4). So, for every $x \in \mathbb{R}$ such that $\cos x \neq 0$ we have

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cdot \cos x - \sin x \cdot (\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

and for every $x \in \mathbb{R}$ such that $\sin x \neq 0$

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{(\cos x)' \cdot \sin x - \cos x \cdot (\sin x)'}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}.$$

Exercise 10.7. Compute derivatives of the following functions:
a) $y = \frac{2x}{1-x^2}$; b) $y = \sqrt[3]{\frac{1+x^2}{1-x^2}}$; c) $y = e^{-x^2 \sin x}$; d) $y = \frac{\sin^2 x}{\sin x^2}$; e) $y = e^x\left(1 + \cot\frac{x}{2}\right)$.

Exercise 10.8. Let $f(x) = x^2$, $x \leq 1$, and $f(x) = ax + b$, $x > 1$. For which $a, b \in \mathbb{R}$ the function f: a) is continuous on $\mathbb{R}$; b) is differentiable on $\mathbb{R}$? Compute f'.

Exercise 10.9. Show that
a) $(\sinh x)' = \cosh x$, $x \in \mathbb{R}$; b) $(\cosh x)' = \sinh x$, $x \in \mathbb{R}$;
c) $(\tanh x)' = \frac{1}{\cosh^2 x}$, $x \in \mathbb{R}$; d) $(\coth x)' = -\frac{1}{\sinh^2 x}$, $x \in \mathbb{R} \setminus \{0\}$.

Exercise 10.10. Let $f(x) = \frac{1}{x^3} e^{-\frac{1}{x^2}}$ for $x \neq 0$ and $f(0) = 0$. Prove that $f'(0) = 0$.