The lecture notes are based on the following literature:

- A.Y. Dorogovtsev, *Mathematical analysis*, Kyiv, Fact, 2004, (Russian).
- K.A. Ross, *Elementary analysis: The theory of calculus*, Undergraduate Texts in Mathematics, Springer New York, 2013.
- I. Lankham, B. Nachtergaele, and A. Schilling, *Linear algebra as an introduction to abstract mathematics*, WSPC, 2016.
- B. P. Demidovič, *Problems and exercises in mathematical analysis*, Moscow, 1997, (Russian).

1 Lecture 1 – Elements of Set Theory and Mathematical Induction

1.1 Elements of Set Theory

The notion of a **set** is one of the most important initial and nondefinable notions of the modern mathematics. By a "set" we will understand any collection into a whole M of definite and separate objects m of our intuition or our thought (Georg Cantor). These objects are called the "elements" of M. Shortly we will use the notation $m \in M$ or $M \ni m$. The fact that m does not belong to M is denoted by $m \notin M$.

A set M can be defined by listing of its elements. For instance,

- $\mathbb{N} = \{1, 2, 3, \dots, n, \dots\}$ — the set of **natural numbers**;
- $\mathbb{Z} = \{\dots, -n, \dots, -1, 0, 1, 2, 3, \dots, n, \dots\}$ — the set of **integer numbers**.

A set also can be defined by specifying of properties of its elements. In any mathematical problem usually consider elements of some quite defined set X. The needed set can be chosen by some property P satisfying the following property: for each x from X either x satisfies P (in this case one writes $P(x)$) or x does satisfy it. This set is denoted by $\{x \in X : P(x)\}$ or $\{x : P(x)\}$. The set which does not contain any elements is called **empty** and is denoted $\emptyset$.

Example 1.1. 1. $\mathbb{N} = \{n \in \mathbb{Z} : n > 0\}$. Here P means "to be positive", which is satisfied by any integer number.

2. Let P denote "to be even". Then $\{2, 4, \dots, 2k, \dots\} = \{n \in \mathbb{N} : P(n)\}$.

3. $\mathbb{Q} = \left\{\frac{m}{n} : n \in \mathbb{N}, m \in \mathbb{Z}\right\}$. This is the set of **rational numbers**.

Exercise 1.1. List elements of the following sets:

a) $\{n \in \mathbb{N} : (n-3)^2 < 7^2\}$;

b) $\left\{n \in \mathbb{N} : \frac{n^2+3n-12}{n} \in \mathbb{N}\right\}$;

c) $\left\{n \in \mathbb{N} : \frac{n+9}{n+1} \in \mathbb{N}\right\}$;

d) $\left\{n \in \mathbb{Z} : n^3 > 10n^2\right\}$.

1.1.1 Operations on Sets

Let A and B be sets.

Definition 1.1. A set A is a **a subset of a set** B, if each element x of A is an element of B (or shortly, $\forall x \in A \ \Rightarrow \ x \in B$). Notation: $A \subset B$.

Definition 1.2 (Operations on sets).

- $A \cup B = \{x : \ x \in A \text{ or } x \in B\}$ — the **union of** A **and** B;
- $A \cap B = \{x : \ x \in A \text{ and } x \in B\}$ — the **intersection of** A **and** B;
- $A \setminus B = \{x : \ x \in A \text{ and } x \notin B\}$ — the **difference of** A **and** B;
- $A \triangle B = \{x : \ x \in A \cup B \text{ and } x \notin A \cap B\}$ — the **symmetric difference of** A **and** B;
- $A^c = \{x \in X : \ x \notin A\}$ — the **complement of** A, where X is some given set containing A.

Exercise 1.2. Show that

a) $A \cup \emptyset = A$, $A \cup A = A$, $A \cup B = B \cup A$, $A \cup (B \cup C) = (A \cup B) \cup C =: A \cup B \cup C$;

b) $A \cap \emptyset = \emptyset$, $A \cap A = A$, $A \cup B = B \cap A$, $A \cap (B \cap C) = (A \cap B) \cap C =: A \cap B \cap C$;

c) $A \triangle B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$, $A \setminus B = A \cap B^c$;

d) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

e) $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.

Let T be a set of indexes and for each $t \in T$ a set A_t is given.

Definition 1.3.

- $\bigcup\limits_{t \in T} A_t = \{x : \ \exists t_0 \in T \ \ A_{t_0} \ni x\}$ — the **union of the family** A_t**,** $t \in T$;
- $\bigcap\limits_{t \in T} A_t = \{x : \ \forall t \in T \ \ A_t \ni x\}$ — the **intersection of the family** A_t**,** $t \in T$;

Example 1.2. Let $A_n = \{1, \ldots, n\}$ for each $n \in \mathbb{N}$. Then

$$\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n = \mathbb{N}, \quad \bigcap_{n \in \mathbb{N}} A_n = \bigcap_{n=1}^{\infty} A_n = \{1\}.$$

1.2 Numbers

1.2.1 Mathematical induction

For more details see [1, Section 1.1].

Let M be a subset of natural numbers which satisfies the following properties

1) $1 \in M$;

2) if $n \in M$, then $n + 1 \in M$.

Then $M = \mathbb{N}$! This is one of the axioms of natural numbers and it is the basis of **mathematical induction**. Let $P_1, P_2, P_3, \ldots$ be a list of statements or propositions that may or may not be true. The principle of mathematical induction asserts all the statements $P_1, P_2, P_3, \ldots$ are true provided

(I1) P_1 is true;

(I2) P_{n+1} is true whenever P_n is true.

We will refer to **(I1)** as the basis for induction and we will refer to **(I2)** as the induction step.

Example 1.3. Prove $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ for positive integers n.
Solution. Our n-th proposition is

$$P_n : \quad 1 + 2 + \cdots + n = \frac{1}{2}n(n+1).$$

Base case: Show that the statement P_n holds for $n = 1$. So,

$$1 = \frac{1}{2} \cdot 1 \cdot (1+1).$$

Induction step: We assume that P_n holds, i.e.

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$$

is true, and must prove P_{n+1}. So,

$$1 + 2 + \cdots + n + (n+1) = \frac{1}{2}n(n+1) + (n+1) = \frac{1}{2}(n+1)(n+2) = \frac{1}{2}(n+1)((n+1)+1).$$

By the principle of mathematical induction, we conclude that P_n is true for all n.

Exercise 1.3. a) Prove that all numbers of the form $5^n - 4n - 1$, $n \in \mathbb{N}$ are divisible by 16.

b) Show that $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$ for each $n \in \mathbb{N}$.

c) Prove the inequality $1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for all $n \in \mathbb{N}$.