Problem sheet 7

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Solutions will be collected during the lecture on Wednesday December 12.

1. **[1x6 points]** Compute derivatives of the following functions:
   a) $f(x) = \frac{1}{4}\ln\frac{x^2-1}{x^2+1}$; b) $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$; c) $f(x) = \ln\tan\frac{x}{2}$;
   d) $f(x) = \arcsin\frac{1-x}{\sqrt{2}}$; e) $f(x) = \arctan\frac{1+x}{1-x}$; f) $f(x) = \sqrt[x]{x}$.

2. **[2 points]** Let a function $f : (a,b) \to \mathbb{R}$ be differentiable on (a,b) and there exists $L \in \mathbb{R}$ such that $|f'(x)| \leq L$ for all $x \in (a,b)$. Show that f is uniformly continuous on (a,b).

3. **[2 points]** Prove the equality
$$3\arccos x - \arccos(3x - 4x^3) = \pi, \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$
   (*Hint:* Compute derivatives of the left and right hand sides of the equality)

4. **[3 points]** Let functions $f, g : (a,b) \to (0,+\infty)$ be differentiable on (a,b) and for every $x \in (a,b)$ $\frac{f'(x)}{f(x)} = \frac{g'(x)}{g(x)}$. Prove that there exists $L > 0$ such that $f(x) = Lg(x)$ for all $x \in (a,b)$.
   (*Hint:* Consider the functions $\ln f$ and $\ln g$)

5. **[3 points]** (*Generalised Bernoulli inequality*) For each $\alpha > 1$, prove that $(1+x)^\alpha \geq 1 + \alpha x$ for all $x > -1$. Moreover, $(1+x)^\alpha = 1 + \alpha x$ iff $x = 0$.

6. **[2x4 points]** Identify the intervals on which the following functions are monotone.
   a) $f(x) = 3x - x^3$, $x \in \mathbb{R}$ b)$f(x) = \frac{2x}{1+x^2}$, $x \in \mathbb{R}$; c) $f(x) = \frac{x^2}{2^x}$, $x \in \mathbb{R}$;
   d) $f(x) = x + \sqrt{|1-x^2|}$, $x \in \mathbb{R}$.