UNIVERSITÄT LEIPZIG

Problem sheet 5

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Solutions will be collected during the lecture on Wednesday November 28.

1. [**1+1 points**] Let a, b be a real numbers, $f(x) = x + 1$, $x \leq 0$ and $f(x) = ax + b$, $x > 0$. a) For which a, b the function f is monotone on $\mathbb{R}$? b) For which a, b the function f is continuous on $\mathbb{R}$?

2. [**1+1 points**] Compute the following limits:
   a) $\lim\limits_{x \to 0}(\tan x - e^x)$; b) $\lim\limits_{x \to 2} \frac{x^2 - 3^x + 1}{x - \sin \pi x}$.

3. [**2 points**] Let $f(x) = \lfloor x \rfloor \sin \pi x$, $x \in \mathbb{R}$. Prove that f is continuous on $\mathbb{R}$ and sketch its graph.
   (*Hint:* If $x \in [k, k+1)$ for some $k \in \mathbb{Z}$, then $\lfloor x \rfloor = k$ and $f(x) = k \sin \pi x$. Find $f(k-)$ and $f(k+)$ at the points k.)

4. [**2 points**] Prove that the function $f(x) = \sin \frac{1}{x}$, $x \neq 0$, and $f(0) = 0$, is discontinuous at 0.

5. [**2x6 points**] Compute the following limits:
   a) $\lim\limits_{x \to 0} \frac{\arcsin x}{x}$; b) $\lim\limits_{x \to 0} (\cos x)^{\frac{1}{x^2}}$; c) $\lim\limits_{x \to +\infty} x(\ln(1+x) - \ln x)$; d) $\lim\limits_{x \to 0} \left(\frac{1 + \sin 2x}{\cos 2x}\right)^{\frac{1}{x}}$;
   e) $\lim\limits_{x \to 0} \frac{\sqrt[7]{\cos x} - 1}{\sqrt[3]{1 + x^2} - 1}$; f) $\lim\limits_{x \to 0} \frac{e^{\sin 2x} - e^{\tan x}}{x}$.

6. [**2 points**] Prove that the function $P(x) = x^3 + 7x^2 - 1$, $x \in \mathbb{R}$, has at least one root, that is, there exists $x_0 \in \mathbb{R}$ such that $P(x_0) = 0$.

7. [**3 points**] Let $f : [a, b] \to \mathbb{R}$ strictly increase on $[a, b]$ and for each $y_0 \in [f(a), f(b)]$ there exist $x_0 \in [a, b]$ such that $f(x_0) = y_0$. Prove that f is continuous on $[a, b]$.

8. [**2 points**] Using the definition, show that the function $f(x) = \sqrt{x}$, $x \in [1, +\infty)$, is uniformly continuous on $[1, +\infty)$.