Problem sheet 12

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Solutions will be collected during the lecture on Wednesday January 30.

1. **[2x4 points]** Investigate the convergence of the following series:
   a) $\sum\limits_{n=1}^{\infty} \frac{3^n n!}{n^n}$; b) $\sum\limits_{n=1}^{\infty} \frac{n^5}{2^n+3^n}$; c) $\sum\limits_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$; d) $\sum\limits_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n(n+1)}$.

2. **[3x2 points]** Investigate the absolute and conditional convergence of the following series:
   a) $\sum\limits_{n=1}^{\infty} (-1)^n \sin^{\alpha} \frac{1}{n}$, where $\alpha > 0$; b) $\sum\limits_{n=1}^{\infty} \frac{\cos n}{n}$.

3. **[2 points]** Show that for each $x \in \mathbb{R}$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots.$$

   (*Hint:* Use Taylor's formula with Lagrangian remainder term (see Theorem 14.1 and Example 14.1) to show that the remainder term converges to 0)

4. **[1x3 points]** Express the following complex numbers in the form $x + yi$ for $x, y \in \mathbb{R}$:
   a) $(2+3i)^2(1+2i)$; b) $\frac{2+3i}{2-i}$; c) $\frac{1}{i} - \frac{1}{(1+i)^2}$.

5. **[2 points]** Compute the real and imaginary parts of $\frac{1}{z^2}$, where $z = x + iy$, $x, y \in \mathbb{R}$.

6. **[2 points]** Compute $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{21}$.

7. **[2x2 points]** Solve the following equations:
   a) $z^2 + z + 3 = 0$; b) $z^3 - i = 0$.

8. **[2 points]** Let $z, w \in \mathbb{C}$. Prove the parallelogram law $|z-w|^2 + |z+w|^2 = 2(|z|^2 + |w|^2)$.