Problem sheet 10

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Solutions will be collected during the lecture on Wednesday January 16.

1. **[3 points]** Using the definition of the integral, prove that the function $f(x) = x$, $x \in [0,1]$, is integrable on $[0,1]$ and compute $\int_0^1 x dx$.

2. **[2 points]** Let $f : [a,b] \to \mathbb{R}$ be a function and $c \in (a,b)$. Show that f is integrable on $[a,b]$, if it is integrable on $[a,c]$ and $[c,b]$.

   (*Hint:* Use the integrability criterion (Theorem 16.2))

3. **[3 points]** Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and g be a non-negative integrable function on $[a,b]$. Show that there exists $\theta \in [a,b]$ such that $\int_a^b f(x)g(x)dx = f(\theta)\int_a^b g(x)dx$.

4. **[2 points]** Let $f : [0,1] \to \mathbb{R}$ be integrable on $[0,1]$. Prove the equality

$$\lim_{n\to\infty} \int_{\frac{1}{n}}^1 f(x)dx = \int_0^1 f(x)dx.$$

5. **[2x5 points]** Compute the following integrals:
   a) $\int_0^{\frac{\pi}{2}} \sin 2x dx$; b) $\int_0^2 |1-x| dx$; c) $\int_0^{2\pi} x^2 \cos x dx$; d) $\int_{-1}^1 \frac{xdx}{\sqrt{5-4x}}$; e) $\int_0^{\ln 2} \sqrt{e^x - 1} dx$.

6. **[3 points]** Compute the area of the region bounded by the graphs of the following functions: $y = x^2$ and $x + y = 2$.

7. **[3 points]** Compute the length of the cycloid, the continuous curve defined by the following functions: $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $t \in [0, 2\pi]$, where $a > 0$.