Mathematical model of coalescing diffusion particles system with variable weights

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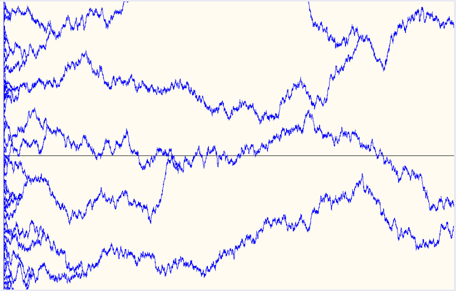
Main object of investigation

System of diffusion particles on the real line that

1. start from some set of points with masses;
2. move independently up to the moment of the meeting;
3. coalesce;
4. have their mass adding after sticking;
5. have their diffusion changed correspondingly to the changing of the mass $(\sigma^2 = \frac{1}{m})$.

Interaction particles systems. Singular interaction

Aratia flow



Arratia R. A. '79

Interaction particles systems. Singular interaction

Arratia flow, mathematical description

$\{x(u,t),\ t \geq 0, u \in \mathbb{R}\}$ such that

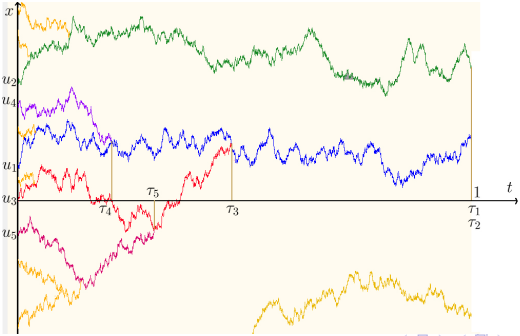
1. $x(u,\cdot)$ is a Brownian motion;
2. $x(u,0) = u, \quad u \in \mathbb{R}$;
3. $x(u,t) \leq x(v,t), \quad u < v, \quad t \geq 0$;
4. $\langle x(u,\cdot), x(v,\cdot)\rangle_t = 0, \quad t < \tau_{u,v}$,
   where $\tau_{u,v} = \inf\{t : x(u,t) = x(v,t)\}$.

Total time of runs of the particles

Let a set of different points $\{u_k, k \in \mathbb{N}\}$ be dense in [0,U].

$$\tau_1 = 1,$$

$$\tau_k = \inf\left\{1; t : \prod_{j=1}^{k-1}(x(u_k,t) - x(u_j,t)) = 0\right\}, \quad k \geq 2.$$



Total time of runs of the particles

Theorem 1

The sum $\sum_{n=1}^{\infty} \tau_n$ is finite a.s. and does not depend on the set $\{u_k, k \in \mathbb{N}\}$.

Theorem 2

For all t the set $\{x(u,t),\ u \in [0,U]\}$ has a finite number of different points.

Dorogovtsev A. A. '04

Special stochastic integral with respect to Arratia flow

Theorem 3

Let $a : \mathbb{R} \to \mathbb{R}$ be a measurable bounded function then the series

$$\sum_{n=1}^{\infty} \int_0^{\tau_n} a(x(u_n, s))dx(u_n, s)$$

is convergent in L_2 and their sum does not depend on the set $\{u_k, k \in \mathbb{N}\}$.

Denote

$$\int_0^U \int_0^{\tau(u)} a(x(u,s))dx(u,s) = \sum_{n=1}^{\infty} \int_0^{\tau_n} a(x(u_n, s))dx(u_n, s).$$

Theorem 4

The random processes $\int_0^U \int_0^{\tau(u)} a(x(u,s))dx(u,s)$, $U \geq 0$, is a $\sigma(x(u,\cdot),\ u \in [0,U])$-martingale.

Dorogovtsev A. A. '06

Girsanov theorem for diffusion processes with coalescence

Let $a : \mathbb{R} \to \mathbb{R}$ be a bounded Lipschitz continuity function

Diffusion flow with coalescing

$\{y(u,t),\ t \geq 0, u \in \mathbb{R}\}$ such that

1. $\mathcal{M}(u,\cdot) = y(u,\cdot) - \int_0^{\cdot} a(y(u,s))ds$ is a Brownian motion, for all $u \in \mathbb{R}$;
2. $y(u,0) = u, \quad u \in \mathbb{R}$;
3. $y(u,t) \leq y(v,t), \quad u < v, \quad t \geq 0$;
4. $\langle \mathcal{M}(u,\cdot), \mathcal{M}(v,\cdot)\rangle_t = 0, \quad t < \sigma_{u,v}$,
   where $\sigma_{u,v} = \inf\{t : y(u,t) = y(v,t)\}$.

Girsanov theorem for diffusion processes with coalescence

Theorem 5

The distribution of y is absolutely continuous with respect to the distribution of x in the space $\mathsf{D}([0,U],\mathsf{C}([0,1]))$ with the density

$$p(x) = \exp\left\{ \int_0^U \int_0^{\tau(u)} a(x(u,s))dx(u,s) - \frac{1}{2}\int_0^U \int_0^{\tau(u)} a(x(u,s))ds \right\}.$$

Dorogovtsev A. A. '07

Asymptotic behaviour

Law of the iterated logarithm for a Wiener process

$$\overline{\lim_{t\to 0}} \frac{|x(u,t)-u|}{\sqrt{2t\ \ln\ln t^{-1}}} = 1 \quad \text{a.s.}$$

Asymptotic in the sup-norm

$$\overline{\lim_{t\to 0}} \sup_{u\in[0,1]} \frac{|x(u,t)-u|}{\sqrt{t\ \ln t^{-1}}} = 1 \quad \text{a.s.}$$

Shamov A. '10

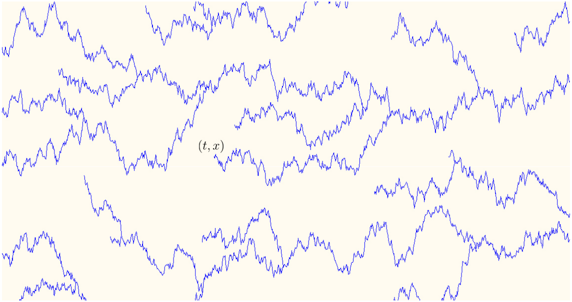
Asymptotic of cluster size

Let $\nu(t) = \lambda\{u: \ x(u,t) = x(0,t)\}, \ t \geq 0$. Then a.s.

$$\underline{\lim_{t\to 0}} \frac{\nu(t)}{\sqrt{2t\ \ln\ln t^{-1}}} \geq 1, \quad \overline{\lim_{t\to 0}} \frac{\nu(t)}{2\sqrt{t\ \ln\ln t^{-1}}} \leq 1.$$

Dorogovtsev A. A., Vovchanskii M. B. '13

Brownian web



Brownian web

System of brownian particles which

1. start from all time-space point in $\mathbb{R} \times \mathbb{R}$;
2. move independently up to the moment of the meeting;
3. coalesce;

Fontes L. R. G., Isopi M., Newman C. M., Ravishankar K. '04

Interaction particles systems. Singular interaction

Coalescing system of non-Brownian particles

System of particles in a metric space such that

1. every particle is described by the Markov process;
2. every two particles move independently up to the moment of the meeting;
3. particles coalesce.

Le Jan Y. '04, Evens S. N. '12

Interaction particles systems. Singular interaction

Main property such systems

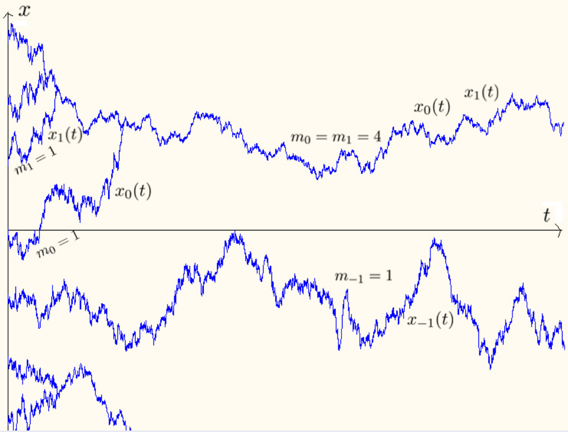
All subsystem of such system may be described as a separate system

Interaction particles systems. Systems concerning particles masses

- Coalescing Brownian particles which have some masses and these masses vary by the some law. The mass does not influence motion of the particles
  Dawson D. A. '01, '04.
- Stochastic differential equation with interaction
  $$\begin{cases} dx(u,t) = a(x(u,t),\mu_t,t)dt + \int_{\mathbb{R}} b(x(u,t),\mu_t,t,q)W(dt,dq) \\ x(u,0) = u, \mu_t = \mu_0 \circ x(\cdot,t)^{-1}. \end{cases}$$
  Dorogovtsev A. A. '07.
- The cases of finite and infinite numbers of particles with sticking that have mass and speed and their motion obey the laws of mass conservation and inertion
  Sinai Ya. G. '96.

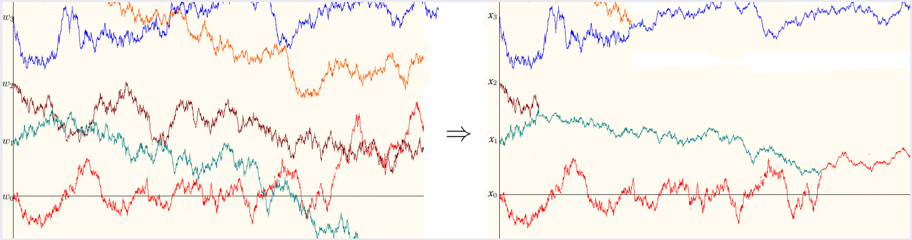
Coalescing particles system with variable weights

$$x_k(0) = k, \quad m_k(0) = 1, \quad k \in \mathbb{Z}$$



Construction of finite system

$\{w_k(t),\ t \geq 0,\ k = -n, \ldots, n\}$ is a system of Wiener processes



$$x^n(\cdot) = F_n(w^n(\cdot)),$$

where $w^n(\cdot) = (-n + w_{-n}(\cdot), \ldots, n + w_n(\cdot))$.

Properties of finite system

Theorem 6

The process $x^n(t),\ t \geq 0$ satisfies the following conditions

1°) $x_k^n(\cdot)$ is a continuous square integrable martingale with respect to the filtration
$$\mathcal{F}_t^n = \sigma(x_i^n(s),\ s \leq t,\ i = -n, \ldots, n);$$

2°) $x_k^n(0) = k,\ k = -n, \ldots, n;$

3°) $x_k^n(t) \leq x_l^n(t),\ k < l,\ t \geq 0;$

4°) $\langle x_k^n(\cdot)\rangle_t = \int\limits_0^t \frac{1}{m_k^n(s)} ds,\ t \geq 0,$
where $m_k^n(t) = |\{i :\ \exists s \leq t\ x_i^n(s) = x_k^n(s)\}|;$

5°) $\langle x_k^n(\cdot), x_l^n(\cdot)\rangle_t = 0, \quad t < \tau_{k,l}^n,$
where $\tau_{k,l} = \inf\{t :\ x_k^n(t) = x_l^n(t)\}.$

Moreover, if a process $y(\cdot)$ satisfies the conditions 1°)–5°) then the processes $y(\cdot)$ and $x^n(\cdot)$ have the same distributions in the space $(\mathsf{C}([0,\infty)))^{2n+1}$.

Some property of a system of Wiener processes

Lemma 7

Let $\{w_k(t),\ t \geq 0,\ k = 0, 1, \ldots\}$ be a system of independent Wiener processes. Define

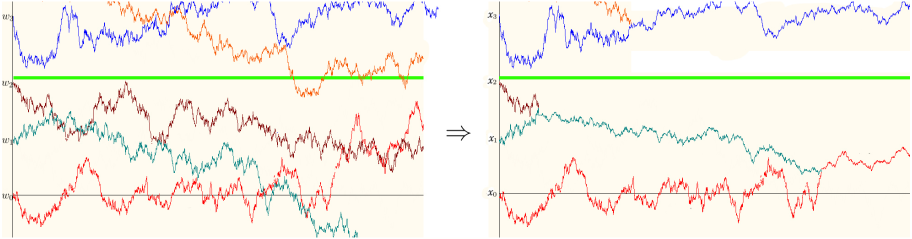
$$\xi_k^T = \max_{t \in [0,T]} \{k + w_k(t)\}, \quad \eta_k^T = \min_{t \in [0,T]} \{k + w_k(t)\}.$$

Then for every $T > 0$ and $\delta \in (0,1)$

$$\mathbb{P}\left\{ \overline{\lim_{n\to\infty}} \left\{ \max_{k=1,\ldots,n} \xi_k^T < n + \delta,\ \eta_{n+1}^T > n + \delta \right\} \right\} = 1.$$

Stabilization

$\{w_k(t),\ t \geq 0,\ k = 0, 1, \ldots\}$ is a fixed system of Wiener processes.



$x^n(\cdot) = F_n(w^n(\cdot)),$
$\mathbb{P}\left\{\exists N\ \forall n \geq N\ \forall t \in [0, T]\ x_k^n(t) = x_k^N(t)\right\} = 1, \quad$ for all $k \in \mathbb{Z}$

The sequence $\{x_k^n(\cdot),\ n \geq k\}$ converges almost surely in $\mathrm{C}([0, \infty))$.

Infinite system

Theorem 8

There exists a sequence of processes $\{x_k(t),\ t \geq 0,\ k \in \mathbb{Z}\}$ such that

1°) $x_k(\cdot)$ is a continuous square integrable martingale with respect to the filtration
$$\mathcal{F}_t = \sigma(x_i(s),\ s \leq t,\ i \in \mathbb{Z});$$

2°) $x_k(0) = k,\ k \in \mathbb{Z};$

3°) $x_k(t) \leq x_l(t),\ k < l,\ t \geq 0;$

4°) $\langle x_k(\cdot)\rangle_t = \int\limits_0^t \frac{1}{m_k(s)} ds,\ t \geq 0,$
where $m_k(t) = |\{i:\ \exists s \leq t\ x_i(s) = x_k(s)\}|;$

5°) $\langle x_k(\cdot), x_l(\cdot)\rangle_t = 0, \quad t < \tau_{k,l},$
where $\tau_{k,l} = \inf\{t:\ x_k(t) = x_l(t)\}.$

Moreover, if a sequence $\{y_k(t),\ t \geq 0,\ k \in \mathbb{Z}\}$ satisfies the conditions $1°$)–$5°$) then $\{y_k(t),\ t \geq 0,\ k \in \mathbb{Z}\}$ and $\{x_k(t),\ t \geq 0,\ k \in \mathbb{Z}\}$ have the same distributions in the space $(\mathrm{C}([0,\infty)))^{\mathbb{Z}}$.

Asymptotic behaviour

$$\overline{\lim_{t\to\infty}} \frac{m_k(t)}{\varphi(t)} = 1;$$

$$\overline{\lim_{t\to\infty}} \frac{|x_k(t)|}{\psi(t)} = 1;$$

Question: What types of functions φ and ψ are?

Estimation of asymptotic growth of the mass

Let $w_1(\cdot)$, $w_2(\cdot)$ be independent Wiener processes and

$$\sigma_{k,l} = \inf\{t : \ k + w_1(t) = l + w_2(t)\}.$$

Properties

1. $\mathbb{P}\{\tau_{k,l} \leq t\} \leq \mathbb{P}\{\sigma_{k,l} \leq t\}$;
2. $\mathbb{P}\left\{\overline{\lim_{t\to+\infty}} \frac{m_k(t)}{4\sqrt{t}\ \ln\ln t} \leq 1\right\} = 1$;
3. $\mathbb{P}\{\tau_{k,k+p} < +\infty\} = 1$.

Asymptotic behaviour of one particle

Using the estimation of asymptotic growth of the mass, the law of the iterated logarithm for a Wiener process and the representation of a square integrable continuous martingale

$$x_k(t) = w(\langle x_k(\cdot)\rangle_t) = w\left(\int_0^t \frac{ds}{m_k(s)}\right)$$

we have

Asymptotic behaviour

1. $\mathbb{P}\left\{\lim_{t\to+\infty} \frac{|x_k(t)|}{\sqrt{2t\,\ln\ln t}} = 0\right\} = 1;$
2. $\mathbb{P}\left\{\overline{\lim_{t\to+\infty}} \frac{|x_k(t)|}{\sqrt[4]{t^{1-\varepsilon}}} = \infty\right\} = 1,$ for all $\varepsilon \in (0,1)$.

Case of general conditions of start points

Theorem 9

For every non-decreasing sequence of real numbers $\{x_i,\ i \in \mathbb{Z}\}$ and sequence of strictly positive real numbers $\{b_i,\ i \in \mathbb{Z}\}$ such that

$$\overline{\lim_{n\to\pm\infty}} \{(x_{n+1} - x_n) \wedge b_{n+1} \wedge b_n\} > 0,$$

there exists a set of processes $\{x_k(t),\ t \geq 0,\ k \in \mathbb{Z}\}$, satisfying

1°) $x_k(\cdot)$ is a continuous square integrable martingale with respect to the filtration

$$\mathcal{F}_t = \sigma(x_i(s),\ s \leq t,\ i \in \mathbb{Z});$$

2°) $x_k(0) = x_k,\ k \in \mathbb{Z};$

3°) $x_k(t) \leq x_l(t),\ k < l,\ t \geq 0;$

4°) $\langle x_k(\cdot)\rangle_t = \int\limits_0^t \frac{1}{m_k(s)} ds,\ t \geq 0,$
where $m_k(t) = \sum\limits_{i \in A_k(t)} b_i,\ \ A_k(t) = \{i \in \mathbb{Z} :\ \exists\ s \leq t,\ x_k(s) = x_i(s)\};$

5°) $\langle x_k(\cdot), x_l(\cdot)\rangle_t = 0, \quad t < \tau_{k,l} = \inf\{t :\ x_k(t) = x_l(t)\}.$

Moreover, the conditions 1°)-5°) uniquely determine the distribution of the process in the space $(\mathsf{C}([0,\infty)))^Z$.

Process of heavy diffusion particles in the space $\mathcal{M}$. Markov property.

Let $\mathcal{M}$ be a set of non-decreasing sequences $(x_k)_{k\in\mathbb{Z}}$ in $\mathbb{R}$ such that

$$\lim_{k\to\infty}\frac{x_k}{k}=1.$$

$$\rho_{\mathcal{M}}((x_k),(y_k))=\max_{k\in\mathbb{Z}}\frac{|x_k-y_k|}{1+|k|}$$

Definition 10

Let $\{x_n(t),\ t\geq 0, n\in\mathbb{Z}\}$ satisfies the conditions 1°)-5°) of Theorem 9 with $b_k=1,\ k\in\mathbb{Z}$. The random process $(x_n(t))_{n\in\mathbb{Z}},\ t\geq 0$, is called the process of heavy diffusion particles in $\mathcal{M}$.

Theorem 11

The process of heavy diffusion particles in $\mathcal{M}$ is a continuous strictly Markov process.

Martingale problem. General conception

Let $\xi(t)$, $t \geq 0$, be a continuous Markov process in a metric space E and $(A, \mathcal{D})$ be a generator of the process $\xi(\cdot)$.
We know

$$\forall f \in \mathcal{D} \quad f(\xi(t)) - f(\xi(0)) - \int_0^t Af(\xi(s))ds \ \text{ is a martingale} \tag{1}$$

and if a process $\eta(t)$, $t \geq 0$, satisfies (1) then $\xi(\cdot) \stackrel{d}{=} \eta(\cdot)$.

Definition 12

A process $\xi(\cdot)$ is called a unique solution of the martingale problem for $(A_1, \mathcal{D}_1)$ if

$$\forall f \in \mathcal{D}_1 \quad f(\xi(t)) - f(\xi(0)) - \int_0^t A_1 f(\xi(s))ds \ \text{ is a martingale} \tag{2}$$

and for every process $\eta(t)$, $t \geq 0$, that satisfies (2), we have $\xi(\cdot) \stackrel{d}{=} \eta(\cdot)$.

Martingale problem for the process of heavy diffusion particles in $\mathcal{M}$

$$\mathfrak{D}_{\mathcal{M}} = \left\{ f \circ \pi_n : \ f \in \mathrm{C}^2(\mathbb{R}^{2n+1}), \ f \text{ has a compact support}, \ n \in \mathbb{N} \right\}.$$

$$\mathfrak{G}_{\mathcal{M}} \widetilde{f}(x) = \frac{1}{2} \sum_{i,j=-n}^{n} \frac{\partial^2}{\partial x_i \partial x_j} f(\pi_n x) \frac{\mathbb{I}_{\{x_i = x_j\}}}{\sum_{l \in \mathbb{Z}} \mathbb{I}_{\{x_i = x_l\}}},$$

where $\widetilde{f} = f \circ \pi_n, \quad \pi_n x = (x_{-n}, \ldots, x_n)$.

Theorem 13

The process of heavy diffusion particles in $\mathcal{M}$ is a unique solution of the martingale problem for $(\mathfrak{G}_{\mathcal{M}}, \mathfrak{D}_{\mathcal{M}})$.

Evolution of particles masses

Let $\mathcal{H}$ be a set of integer valued measures μ on the real line such that

$$\lim_{n\to\infty} \frac{\mu([0,n))}{n} = 1, \quad \lim_{n\to\infty} \frac{\mu([-n,0))}{n} = 1. \tag{3}$$

where $\rho_{\mathcal{H}} = \rho_1 + \rho_2$,
ρ_1 is a metric of week convergence on bounded intervals and
ρ_2 is a uniform distance between the elements of the sequence (3).

Definition 14

Let $\{x_n(t),\ t \geq 0, n \in \mathbb{Z}\}$ satisfies the conditions 1°)-5°) of Theorem 9 with $b_k = 1,\ k \in \mathbb{Z}$. The random process $\sum_{k\in\mathbb{Z}} \delta_{x_k(t)}, \quad t \geq 0$, is called the process of heavy diffusion particles in $\mathcal{H}$.

Theorem 15

The process of heavy diffusion particles in $\mathcal{H}$ is a continuous strictly Markov process.

View of the generator

$$\mathfrak{G}_{\mathcal{H}} F(\mu) = \frac{1}{2} \iint\limits_{\mathbb{R}^2} \frac{\partial^2}{\partial x \partial y} \frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)} \delta_x(dy)\mu(dx) +$$

$$+\frac{1}{2} \int\limits_{\mathbb{R}} \frac{d^2}{dx^2} \frac{\delta F(\mu)}{\delta\mu(x)} \mu^*(dx),$$

$$\mu^* = \sum_{y \in \operatorname{supp} \mu} \delta_y,$$

$$\frac{\delta F(\mu)}{\delta\mu(x)} = \lim_{\varepsilon \to 0+} \frac{F(\mu + \varepsilon\delta_x) - F(\mu)}{\varepsilon}.$$

Domain of the generator

$$F_{\varphi,m}(\mu) = \int \dots \int \varphi(x_1, \dots, x_m)\mu(dx_1)\dots\mu(dx_m),$$

$$\mathfrak{D}_{\mathcal{H}} = \mathsf{span}\{F_{\varphi,m} : \ \varphi \in \Phi_m, \ m \in \mathbb{N}\},$$

where Φ_m is a set of functions $\varphi : \mathbb{R}^m \to \mathbb{R}$ such that

- φ is continuous,
- φ has compact support,
- φ is symmetric,
- φ satisfies some boundary conditions.

Definition of martingale problem

Definition 16

$\{\mu_t,\ t \geq 0\}$ is a solution of the martingale problem for $(\mathfrak{G}_{\mathcal{H}}, \mathfrak{D}_{\mathcal{H}})$ if

1) for every function $F \in \mathfrak{D}_{\mathcal{H}}$,

$$F(\mu_t) - F(\mu_0) - \int_0^t \mathfrak{G}_{\mathcal{H}}(F(\mu_s))ds \text{ is a martingale;}$$

2) there exists a continuous strictly Markov process $\{(x_k(t))_{k\in\mathbb{Z}},\ t \geq 0\}$ in $\mathcal{M}$ such that

$$\mu_t = \sum_{k\in\mathbb{Z}} \delta_{x_k(t)}, \quad t \geq 0;$$

3) for each $k \in \mathbb{Z}$ and any function $f \in C^2([0, +\infty))$ that is bounded together with its derivatives and satisfies the condition $f''(0) = 0$, the difference

$$f(x_{k+1}(t) - x_k(t)) - \frac{1}{2}\int_0^t f''(x_{k+1}(s) - x_k(s))\left[\frac{1}{\sqrt{m_{k+1}(s)}} + \frac{1}{\sqrt{m_k(s)}}\right]ds$$

is a martingale.

Martingale problem

Theorem 17.

The process of heavy diffusion particles in $\mathcal{H}$ is a unique solution of the martingale problem for $(\mathfrak{G}_{\mathcal{H}}, \mathfrak{D}_{\mathcal{H}})$.

Problem

Question: Whether there exists a heavy diffusion particles system which start from all points of the real line?

We want to construct a system of processes $\{x(u,t),\ t \geq 0,\ u \in \mathbb{R}\}$ such that

1°) $x(u,\cdot)$ is a continuous square integrable martingale with respect to the filtration
$$\mathcal{F}_t = \sigma(x(u,s),\ s \leq t,\ u \in \mathbb{R});$$

2°) $x(u,0) = u,\ u \in \mathbb{R};$

3°) $x(u,t) \leq x(v,t),\ u < v,\ t \geq 0;$

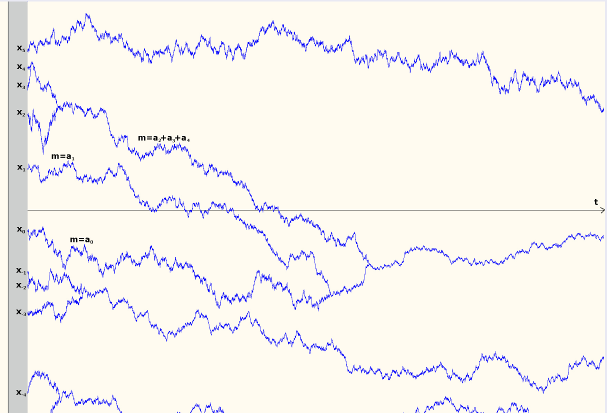
4°) $\langle x(u,\cdot)\rangle_t = \int\limits_0^t \frac{1}{m(u,s)} ds,\ t \geq 0,$
where $m(u,t) = \lambda\{v:\ x(v,t) = x(u,t)\};$

5°) $\langle x(u,\cdot), x(v,\cdot)\rangle_t = 0, \quad t < \tau_{u,v} = \inf\{t:\ x(u,t) = x(v,t)\}.$

Stationary case

We suppose that

1. there is a finite number of particles on each interval.
2. the mass distribution of the particles at the moment of start has stationary distribution with respect to a spatial variable



Stationary measure with respect to a spatial variable

Let $\mathfrak{N}$ be a set of point measures μ on $\mathbb{R}$ which have a finite number of atoms on every interval.

Definition 18

A measure μ on $\mathbb{R}$ is called a stationary point measure with respect to a spatial variable if μ is a map from $\mathcal{B}(\mathbb{R}) \times \Omega$ to $[0, \infty]$ that satisfies the following conditions

1) for each $B \in \mathcal{B}(\mathbb{R})$, $\mu(B, \cdot)$ is a random variable;
2) $\mu(\cdot, \omega) \in \mathfrak{N}$, for all $\omega \in \Omega$;
3) for any $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})$ and $h \in \mathbb{R}$,

$$(\mu(B_1), \ldots, \mu(B_n)) \stackrel{d}{=} (\mu(B_1 + h), \ldots, \mu(B_n + h)).$$

Existence of mathematical model

Theorem 19

Let $\mu = \sum\limits_{k\in\mathbb{Z}} a_k \delta_{x_k}$ be a stationary point measure with respect to a spatial variable on $\mathbb{R}$ and $\mu([0,\infty]) \neq 0$. Then there exists a system of processes $\{x_k(t),\ k\in\mathbb{Z},\ t\geq 0\}$ such that

1°) $x_k(\cdot) - x_k$ is a continuous local martingale with respect to $(\mathcal{F}_t)_{t\geq 0} = (\sigma(x_k(s),\ s\leq t,\ k\in\mathbb{Z}))_{t\geq 0}$;

2°) $x_k(0) = x_k,\ k\in\mathbb{Z}$;

3°) $x_l(t) \leq x_k(t)$, for all $l<k$;

4°) the quadratic characteristic $\langle x_k(\cdot) - x_k\rangle_t = \int\limits_0^t \frac{ds}{m_k(s)}$;

5°) the joint characteristic $\langle x_l(\cdot) - x_l, x_k(\cdot) - x_k\rangle_t = 0, \quad t<\tau_{l,k}$.

Stationary case

Corollary 20

For each $t \geq 0$, $\mu_t = \sum_{k \in \mathbb{Z}} a_k \delta_{x_k(t)}$ is stationary measure with respect to a spatial variable.

Coalescing diffusion particles with drift

System of diffusion particles on the real line that

1. start from some set of points with masses;
2. move independently up to the moment of the meeting;
3. coalesce;
4. have their mass adding after sticking;
5. have their diffusion changed correspondingly to the changing of the mass;
6. have evolution of particle described by SDE

$$dx(t) = \frac{a(x(t))}{m(t)}dt + \frac{\sigma(x(t))}{\sqrt{m(t)}}dw(t).$$

Existence of the mathematical model

Theorem 21

Let a, σ be bounded Lipschitz continuity functions and $\inf\limits_{x\in\mathbb{R}} \sigma(x) > 0$. Then for every non-decreasing sequence of real numbers $\{x_i,\ i \in \mathbb{Z}\}$ and sequence of strictly positive real numbers $\{b_i,\ i \in \mathbb{Z}\}$ such that

$$\overline{\lim_{n\to\pm\infty}} \{(x_{n+1} - x_n) \wedge b_{n+1} \wedge b_n\} > 0,$$

there exists a set of processes $\zeta_i(t),\ t \geq 0,\ i \in \mathbb{Z}$, satisfying

1°) $\mathfrak{M}_i = \zeta_i(\cdot) - \int\limits_0^{\cdot} \frac{a(\zeta_i(s))}{m_i(s)} ds$ *is a continuous square integrable martingale with respect to the filtration* $\mathcal{F}_t^{\zeta} = \sigma(\zeta_i(s),\ s \leq t,\ i \in \mathbb{Z})$, *where* $m_i(t) = \sum\limits_{j \in A_i(t)} b_j$,
$A_i(t) = \{j :\ \exists s \leq t\ \zeta_j(s) = \zeta_i(s)\}$;

2°) $\zeta_i(0) = x_i,\ i \in \mathbb{Z}$;

3°) $\zeta_i(t) \leq \zeta_j(t),\ i < j,\ t \geq 0$;

4°) $\langle \mathfrak{M}_i \rangle_t = \int\limits_0^t \frac{\sigma^2(\zeta_i(s))}{m_i(s)} ds,\ t \geq 0$;

5°) $\langle \mathfrak{M}_i, \mathfrak{M}_j \rangle_t = 0, \quad t < \tau_{i,j} = \inf\{t : \zeta_i(t) = \zeta_j(t)\}$.

Unique distribution of probability

Theorem 22

The conditions 1°*)-*5°*) of Theorem 21 uniquely determine the distribution of the process in the space* $(\mathsf{C}([0,\infty)))^{\mathbb{Z}}$

Markov property

$$b \in (0,\infty)^{\mathbb{Z}}$$

$$\varlimsup_{n\to\pm\infty} \{b_n \wedge b_{n+1}\} > 0$$

$$\mathrm{K}^b = \left\{ x \in \mathbb{R}^{\mathbb{Z}} : \varlimsup_{n\to\pm\infty} \{(x_{n+1} - x_n) \wedge b_{n+1} \wedge b_n\} > 0 \right\}$$

$$\mathbb{P}_x^{\zeta} = \mathbb{P} \circ \zeta^{-1}, \quad \zeta(0) = x$$

Theorem 23

The set of the distributions $\{\mathbb{P}_x^{\zeta},\ x \in \mathrm{K}^b\}$ is a strictly Markov system.

Estimation of asymptotic growth of the mass

In case $b_k = 1$, $x_{k+1} - x_k > \delta$, $k \in \mathbb{Z}$

$$\mathbb{P}\left\{ \overline{\lim_{t \to +\infty}} \frac{\delta m_k(t)}{8\|\sigma\| \sqrt{t \ln \ln t}} \leq 1 \right\} = 1.$$

Open problems

1. Strict estimation of the probability $\mathbb{P}\{\tau_{0,n} \geq t\}$ for large n and t;
2. Asymptotic growth of the particle mass;
3. Asymptotic behaviour of the particle;
4. Start from all points of the real line.

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Thank you!