A particle model for Wasserstein type diffusion

Vitalii Konarovskyi

Leipzig University

Heriot-Watt University, Edinburgh – 2019

vitalii.konarovskyi@math.uni-leipzig.de

UNIVERSITÄT LEIPZIG

Motivation: Dean-Kawasaki equation

$$\partial_t \mu_t = \frac{\beta}{2}\Delta\mu_t + \nabla\cdot\left(\mu_t \nabla \frac{\delta F(\mu_t)}{\delta \mu_t}\right) + \nabla\cdot[\sqrt{\mu_t}\dot{W}_t]$$

– Dean-Kawasaki equation

The equation appears in macroscopic fluctuation theory and glass dynamic models

D. Dean '96, K. Kawasaki, E. Vanden-Eijnden, A. Donev, B. Derrida, J. Zimmer, ...

Meaning of solutions: A measure-valued process μ_t solves the equation if for each smooth bdd φ $\langle\varphi,\mu_t\rangle = \int\varphi(x)\mu_t(dx)$ satisfies:

$$\langle\varphi,\mu_t\rangle - \int_0^t \left(\frac{\beta}{2}\langle\Delta\varphi,\mu_s\rangle + \left\langle\nabla\varphi,\nabla\frac{\delta F(\mu_t)}{\delta\mu_t}\right\rangle\right) ds$$

martingale with quadratic variation $\int_0^t \langle|\nabla\varphi|^2,\mu_s\rangle ds$

Motivation: Dean-Kawasaki equation

$$\partial_t \mu_t = \frac{\beta}{2}\Delta \mu_t + \nabla \cdot \left(\mu_t \nabla \frac{\delta F(\mu_t)}{\delta \mu_t} \right) + \nabla \cdot [\sqrt{\mu_t}\dot{W}_t]$$

The

D. I

M

equ

$$dx_i(t) = \sum_{j=1}^{n} \nabla V(x_i(t) - x_j(t))dt + dw_i(t), \quad i = 1, \ldots, n,$$

where w_i are independent Wiener processes

$$\mu_t = \sum_{i=1}^{n} \delta_{x_i(t)} \quad + \text{formal using of the Ito formula}$$

$$F(\mu) = \iint V(x-y)\mu(dx)\mu(dy)$$

martingale with quadratic variation $\int_0 \langle |\nabla \varphi|^2, \mu_s \rangle ds$

Motivation: Dean-Kawasaki equation

$$\partial_t \mu_t = \frac{\beta}{2}\Delta \mu_t + \nabla \cdot \left(\mu_t \nabla \frac{\delta F(\mu_t)}{\delta \mu_t} \right) + \nabla \cdot [\sqrt{\mu_t}\dot{W}_t]$$

$$dx_i(t) = \sum_{j=1}^{n} \nabla V(x_i(t) - x_j(t))dt + dw_i(t), \quad i = 1, \dots, n,$$

where w_i are independent Wiener processes

$$\mu_t = \sum_{i=1}^{n} \delta_{x_i(t)} \qquad + \text{formal using of the Ito formula}$$

$$F(\mu) = \iint V(x-y)\mu(dx)\mu(dy)$$

martingale with quadratic variation

Motivation: Dean-Kawasaki equation

$$\partial_t \mu_t = \frac{\beta}{2}\Delta\mu_t + \nabla\cdot\left(\mu_t \nabla \frac{\delta F(\mu_t)}{\delta \mu_t}\right) + \nabla\cdot[\sqrt{\mu_t}\dot{W}_t]$$

The

D. I

M

equ

$$dx_i(t) = \sum_{j=1}^{n} \nabla V(x_i(t) - x_j(t))dt + dw_i(t), \quad i = 1, \ldots, n,$$

where w_i are independent Wiener processes

$$\mu_t = \sum_{i=1}^{n} \delta_{x_i(t)} \qquad + \text{formal using of the Ito formula}$$

$$F(\mu) = \iint V(x-y)\mu(dx)\mu(dy)$$

martingale with quadratic variation

Motivation: Dean-Kawasaki equation

$$\partial_t \mu_t = \frac{\beta}{2}\Delta\mu_t + \nabla\cdot\left(\mu_t \nabla \frac{\delta F(\mu_t)}{\delta \mu_t}\right) + \nabla\cdot[\sqrt{\mu_t}\dot{W}_t]$$

– Dean-Kawasaki equation

The equation appears in macroscopic fluctuation theory and glass dynamic models

D. Dean '96, K. Kawasaki, E. Vanden-Eijnden, A. Donev, B. Derrida, J. Zimmer, ...

Meaning of solutions: A measure-valued process μ_t solves the equation if for each smooth bdd φ $\langle\varphi,\mu_t\rangle = \int\varphi(x)\mu_t(dx)$ satisfies:

$$\langle\varphi,\mu_t\rangle - \int_0^t \left(\frac{\beta}{2}\langle\Delta\varphi,\mu_s\rangle + \left\langle\nabla\varphi, \nabla\frac{\delta F(\mu_t)}{\delta\mu_t}\right\rangle\right) ds$$

martingale with quadratic variation $\int_0^t \langle|\nabla\varphi|^2,\mu_s\rangle ds$

Motivation: Dean-Kawasaki equation

$$\partial_t \mu_t = \frac{\beta}{2}\Delta \mu_t + \nabla \cdot \left(\mu_t \nabla \frac{\delta F(\mu_t)}{\delta \mu_t} \right) + \nabla \cdot [\sqrt{\mu_t} \dot{W}_t]$$

– Dean-Kawasaki equation

The equation appears in macroscopic fluctuation theory and glass dynamic models

D. Dean '96, K. Kawasaki, E. Vanden-Eijnden, A. Donev, B. Derrida, J. Zimmer, ...

Meaning of solutions: A measure-valued process μ_t solves the equation if for each smooth bdd φ $\langle \varphi, \mu_t \rangle = \int \varphi(x) \mu_t(dx)$ satisfies:

$$\langle \varphi, \mu_t \rangle - \int_0^t \left(\frac{\beta}{2} \langle \Delta \varphi, \mu_s \rangle + \left\langle \nabla \varphi, \nabla \frac{\delta F(\mu_t)}{\delta \mu_t} \right\rangle \right) ds$$

martingale with quadratic variation $\int_0^t \langle |\nabla \varphi|^2, \mu_s \rangle ds$

Motivation: Dean-Kawasaki equation

$$\partial_t \mu_t = \frac{\beta}{2}\Delta\mu_t + \nabla\cdot\left(\mu_t \nabla \frac{\delta F(\mu_t)}{\delta \mu_t}\right) + \nabla\cdot[\sqrt{\mu_t}\dot{W}_t]$$

– Dean-Kawasaki equation

The equation appears in macroscopic fluctuation theory and glass dynamic models

D. Dean '96, K. Kawasaki, E. Vanden-Eijnden, A. Donev, B. Derrida, J. Zimmer, ...

Meaning of solutions: A measure-valued process μ_t solves the equation if for each smooth bdd φ $\langle\varphi,\mu_t\rangle = \int\varphi(x)\mu_t(dx)$ satisfies:

$$\langle\varphi,\mu_t\rangle - \int_0^t \left(\frac{\beta}{2}\langle\Delta\varphi,\mu_s\rangle + \left\langle\nabla\varphi,\nabla\frac{\delta F(\mu_t)}{\delta\mu_t}\right\rangle\right) ds$$

martingale with quadratic variation $\int_0^t \langle|\nabla\varphi|^2,\mu_s\rangle ds$

Motivation: Dean-Kawasaki equation

$$\partial_t \mu_t = \frac{\beta}{2}\Delta\mu_t + \nabla\cdot\left(\mu_t \nabla \frac{\delta F(\mu_t)}{\delta \mu_t}\right) + \nabla\cdot[\sqrt{\mu_t}\dot{W}_t]$$

– Dean-Kawasaki equation

The equation appears in macroscopic fluctuation theory and glass dynamic models

D. Dean '96, K. Kawasaki, E. Vanden-Eijnden, A. Donev, B. Derrida, J. Zimmer, ...

Meaning of solutions: A measure-valued process μ_t solves the equation if for each smooth bdd φ $\langle \varphi, \mu_t\rangle = \int \varphi(x)\mu_t(dx)$ satisfies:

$$\langle \varphi, \mu_t\rangle - \int_0^t \left(\frac{\beta}{2}\langle \Delta\varphi, \mu_s\rangle + \left\langle \nabla\varphi, \nabla \frac{\delta F(\mu_t)}{\delta \mu_t}\right\rangle\right) ds$$
$$= \int_0^t \sqrt{\langle |\nabla\varphi|^2, \mu_s\rangle} dw_\varphi(s)$$

martingale with quadratic variation $\int_0^t \langle |\nabla \varphi|^2, \mu_s\rangle ds$

Ill-Posedness vs. triviality

$$\partial_t \mu_t = \frac{\beta}{2}\Delta\mu_t + \nabla\cdot\left(\mu_t \nabla \frac{\delta F(\mu_t)}{\delta \mu_t}\right) + \nabla\cdot[\sqrt{\mu_t}\dot{W}_t]$$

Theorem. K., Lehmann, von Renesse (Elect. Comm. Probab '19)
Let $F = 0$ and $\mu_0(\mathbb{R}^d) = 1$. Then a solution μ_t to the DK equation only exists (and is unique) for $\beta = n \in \mathbb{N}$ and

$$\mu_t = \frac{1}{n}\sum_{i=1}^{n} \delta_{w_i(t)},$$

where w_i are independent Brownian motions with diffusion rate n.

Even F is "smooth" then the DK equation also has a solution only for $\beta = n \in \mathbb{N}$ and it has a similar form.

K., Lehmann, von Renesse (arXiv:1812.11068)

Ill-Posedness vs. triviality

$$\partial_t \mu_t = \frac{\beta}{2}\Delta\mu_t + \nabla\cdot\left(\mu_t \nabla \frac{\delta F(\mu_t)}{\delta \mu_t}\right) + \nabla\cdot[\sqrt{\mu_t}\dot{W}_t]$$

Theorem. K., Lehmann, von Renesse (Elect. Comm. Probab '19)
Let $F = 0$ and $\mu_0(\mathbb{R}^d) = 1$. Then a solution μ_t to the DK equation only exists (and is unique) for $\beta = n \in \mathbb{N}$ and

$$\mu_t = \frac{1}{n}\sum_{i=1}^{n} \delta_{w_i(t)},$$

where w_i are independent Brownian motions with diffusion rate n.

Even F is "smooth" then the DK equation also has a solution only for $\beta = n \in \mathbb{N}$ and it has a similar form.

K., Lehmann, von Renesse (arXiv:1812.11068)

Ill-Posedness vs. triviality

$$\partial_t \mu_t = \frac{\beta}{2}\Delta\mu_t + \nabla\cdot\left(\mu_t \nabla \frac{\delta F(\mu_t)}{\delta\mu_t}\right) + \nabla\cdot[\sqrt{\mu_t}\dot{W}_t]$$

For which Γ the equation

$$\partial_t \mu_t = \frac{\beta}{2}\Delta\mu_t + \Gamma(\mu_t) + \nabla\cdot[\sqrt{\mu_t}\dot{W}_t]$$

has non-trivial physically available solutions

where w_i are independent Brownian motions with diffusion rate n.

Even F is "smooth" then the DK equation also has a solution only for $\beta = n \in \mathbb{N}$ and it has a similar form.

K., Lehmann, von Renesse (arXiv:1812.11068)

Varadhan formula and Wasserstein space

Let $\mathcal{P}_2(\mathbb{R}^d)$ denote the space of probability measures on $\mathbb{R}^d$ with

$$\int |x|^2 \mu(dx) < \infty.$$

$$d_{\mathcal{W}}(\mu,\nu) = \left(\inf\left\{\mathbb{E}|\xi-\eta|^2 :\ \xi \sim \mu, \eta \sim \nu\right\}\right)^{\frac{1}{2}}$$

– Wasserstein distance on $\mathcal{P}_2(\mathbb{R}^d)$

There is known a "singular" Γ such that the DK equation has a solution μ_t on $\mathcal{P}_2([0,1])$ called the **Wasserstein diffusion** that is a Markov process and satisfies the Varadhan formula

$$\mathbb{P}\{\mu_t = \nu\} \sim e^{-\frac{d^2_{\mathcal{W}}(\mu_0,\nu)}{2t}}, \quad t \ll 1$$

von Renesse, Sturm (Ann. Probab. '09)

Varadhan formula and Wasserstein space

Let $\mathcal{P}_2(\mathbb{R}^d)$ denote the space of probability measures on $\mathbb{R}^d$ with

$$\int |x|^2 \mu(dx) < \infty.$$

$$d_{\mathcal{W}}(\mu, \nu) = \left(\inf\left\{\mathbb{E}|\xi - \eta|^2 : \ \xi \sim \mu, \eta \sim \nu\right\}\right)^{\frac{1}{2}}$$

– Wasserstein distance on $\mathcal{P}_2(\mathbb{R}^d)$

There is known a "singular" Γ such that the DK equation has a solution μ_t on $\mathcal{P}_2([0,1])$ called the **Wasserstein diffusion** that is a Markov process and satisfies the Varadhan formula

$$\mathbb{P}\{\mu_t = \nu\} \sim e^{-\frac{d^2_{\mathcal{W}}(\mu_0, \nu)}{2t}}, \quad t \ll 1$$

von Renesse, Sturm (Ann. Probab. '09)

Varadhan formula and Wasserstein space

Let $\mathcal{P}_2(\mathbb{R}^d)$ denote the space of probability measures on $\mathbb{R}^d$ with

$$\int |x|^2 \mu(dx) < \infty.$$

$$d_{\mathcal{W}}(\mu,\nu) = \left(\inf \left\{\mathbb{E}|\xi - \eta|^2 : \ \xi \sim \mu, \eta \sim \nu\right\}\right)^{\frac{1}{2}}$$

– Wasserstein distance on $\mathcal{P}_2(\mathbb{R}^d)$

There is known a "singular" Γ such that the DK equation has a solution μ_t on $\mathcal{P}_2([0,1])$ called the **Wasserstein diffusion** that is a Markov process and satisfies the Varadhan formula

$$\mathbb{P}\{\mu_t = \nu\} \sim e^{-\frac{d^2_{\mathcal{W}}(\mu_0,\nu)}{2t}}, \quad t \ll 1$$

von Renesse, Sturm (Ann. Probab. '09)

Varadhan formula and Wasserstein space

Let $\mathcal{P}_2(\mathbb{R}^d)$ denote the space of probability measures on $\mathbb{R}^d$ with

Aim: We are going to construct non-trivial solutions of the equation

$$\partial_t \mu_t = \frac{\beta}{2}\Delta \mu_t + \Gamma(\mu_t) + \nabla \cdot [\sqrt{\mu_t}\dot{W}_t]$$

and show that it satisfies the Varadhan formula

$$\mathbb{P}\{\mu_t = \nu\} \sim e^{-\frac{d^2_{\mathcal{W}}(\mu_0,\nu)}{2t}}, \quad t \ll 1$$

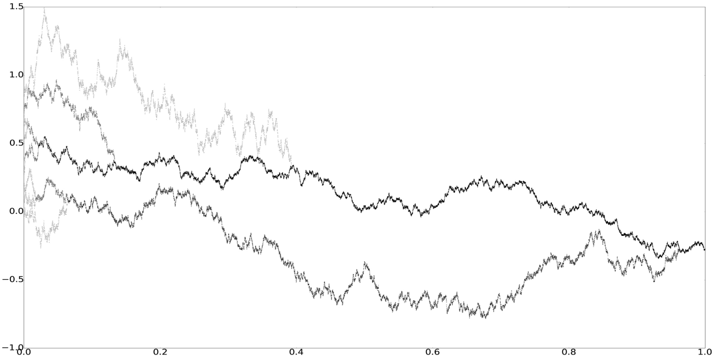
Tool: Particle approach

von Renesse, Sturm (Ann. Probab. '09)

Modified Arratia flow on $\mathbb{R}$

Consider the system of diffusion particles on the real line such that

1. particles move independently and coalesce after meeting
2. each particle has a mass that obeys the conservation law
3. diffusion rate of each particle inversely proportional to its mass



Grayscale colour coding is for particle mass

Modified Arratia flow on $\mathbb{R}$

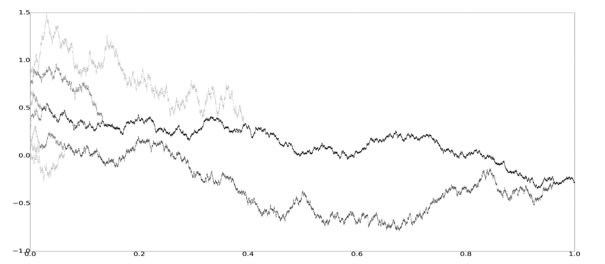
Co

1

2

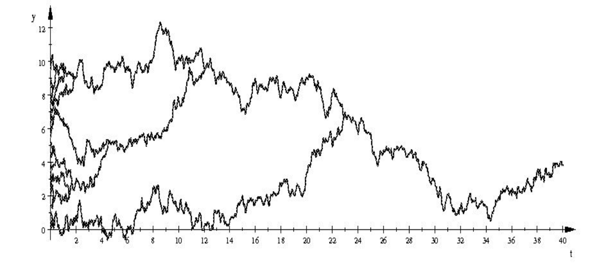
3

Our system



Grayscale colour coding is for particle mass

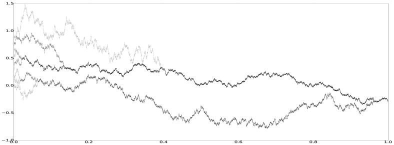
The classical Arratia flow (Brownian web)



Modified Arratia flow on $\mathbb{R}$

Consider the system of diffusion particles on the real line such that

1. particles move independently and coalesce after meeting
2. each particle has a mass that obeys the conservation law
3. diffusion rate of each particle inversely proportional to its mass



Theorem. K. (Ann. Probab. '17 and Electron. J. Probab. '17)
There exists a system of processes $X(u,t)$, $t \geq 0$, $u \in [0,1]$, such that

1. $X(u,0) = u$
2. $X(u,t) \leq X(v,t) \quad u < v$
3. $X(u,\cdot)$ is continuous matringale
4. joint quadratic variation is $d\left\langle X(u,\cdot), X(v,\cdot)\right\rangle_t = \frac{\mathbb{I}_{\{X(u,t)=X(v,t)\}}}{m(u,t)} dt$,

   where $\pi(u,s) = \{v : \ X(u,t) = X(v,t)\}$, $\quad m(u,s) = \mathrm{Leb}\, \pi(u,s)$

Modified Arratia flow and Varadhan formula

Theorem. K., von Renesse (Comm. Pure Appl. Math. '19)
The process $\mu_t = X(\cdot, t)|_\# \mathrm{Leb}$ that describes the evolution of particle mass

- solves

$$\partial \mu_t = \frac{1}{2}\Delta \mu_t^* + \nabla \cdot (\sqrt{\mu_t}\dot{W}_t), \quad \text{in } \mathcal{P}_2(\mathbb{R}),$$

  where $\mu_t^* = \sum_{x \in \operatorname{supp} \mu_t} \delta_x$
- satisfies the Varadhan formula

$$\mathbb{P}\{\mu_t = \nu\} \sim e^{-\frac{d^2_{\mathcal{W}}(\mathrm{Leb}_{[0,1]}, \nu)}{2t}}, \quad t \ll 1$$

Reversible case:

1. Particles move independently and sticky-reflect from each other
2. each particle has a mass that obeys the conservation law
3. diffusion rate of each particle inversely proportional to its mass

Theorem. K. '18
There exists a system of processes $X(u,t)$, $t \geq 0$, $u \in [0,1]$, such that

1. $X(u,0) = u$
2. $X(u,t) \leq X(v,t) \quad u < v$
3. $X(u,t) - \int_0^t \left(u - \frac{1}{m(u,s)} \int_{\pi(u,s)} v dv \right) ds$ is continuous matringale
4. joint quadratic variation is $\langle X(u,\cdot), X(v,\cdot) \rangle_t = \int_0^t \frac{\mathbb{I}_{\{X(u,s)=X(v,s)\}}}{m(u,s)} ds$,

   where $\pi(u,s) = \{v : \ X(u,t) = X(v,t)\}, \quad m(u,s) = \mathrm{Leb}\, \pi(u,s)$

Reversible case:

1. Particles move independently and sticky-reflect from each other
2. each particle has a mass that obeys the conservation law
3. diffusion rate of each particle inversely proportional to its mass

Theorem. K. '18
There exists a system of processes $X(u,t)$, $t \geq 0$, $u \in [0,1]$, such that

1. $X(u,0) = u$
2. $X(u,t) \leq X(v,t) \quad u < v$
3. $X(u,t) - \int_0^t \left(u - \frac{1}{m(u,s)} \int_{\pi(u,s)} v dv\right) ds$ is continuous matringale
4. joint quadratic variation is $\langle X(u,\cdot), X(v,\cdot)\rangle_t = \int_0^t \frac{\mathbb{I}_{\{X(u,s)=X(v,s)\}}}{m(u,s)} ds$,

where $\pi(u,s) = \{v : \ X(u,t) = X(v,t)\}, \quad m(u,s) = \mathrm{Leb}\, \pi(u,s)$

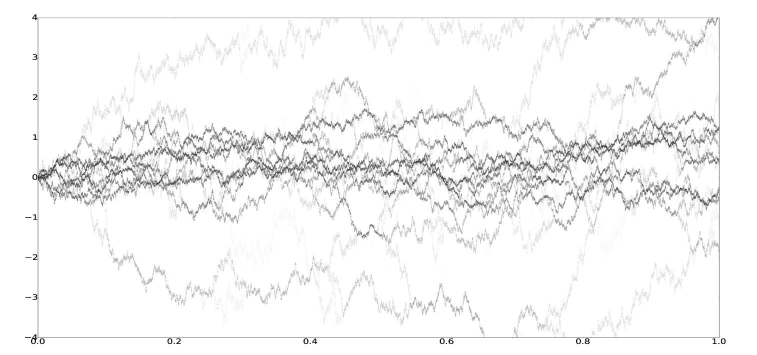
C

Re

1

2

3

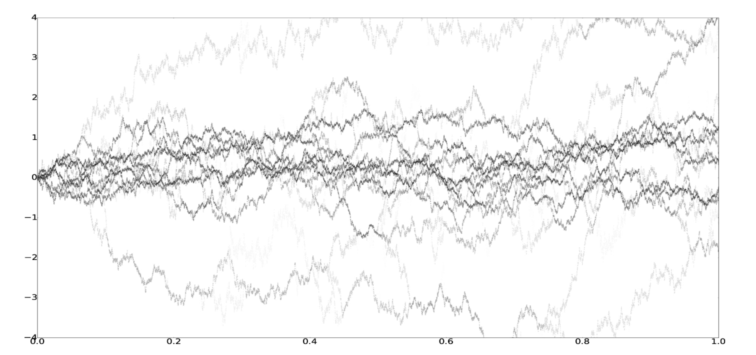


Grayscale colour coding is for particle mass

It is a physical improvement of the Howitt-Warren flow

4. joint quadratic variation is $\langle X(u,\cdot), X(v,\cdot)\rangle_t = \int_0^t \frac{\mathbb{I}_{\{X(u,s)=X(v,s)\}}}{m(u,s)}\,ds$,

   where $\pi(u,s) = \{v: \ X(u,t) = X(v,t)\}, \quad m(u,s) = \mathrm{Leb}\,\pi(u,s)$



Grayscale colour coding is for particle mass

It is a physical improvement of the Howitt-Warren flow

Let $N(t)$ denote the number of distinct particles at time t

- $\int_0^T N(t)dt < \infty$ a.s. $\forall T > 0$
- $\{t : N(t) = \infty\}$ is dense in $[0, \infty)$ a.s. K. '19

Theorem. K., von Renesse '18
There exists a σ-finite full supported measure Ξ on $\mathcal{P}_2(\mathbb{R})$ such that

- Ξ is invariant for $\mu_t = X(\cdot,t)|_\# \operatorname{Leb}$
- μ_t, that describes the evolution of particle mass, solves

$$\partial \mu_t = \frac{1}{2}\Delta \mu_t^* + \nabla \cdot (\sqrt{\mu_t}\dot{W}_t), \quad \text{in } \mathcal{P}_2(\mathbb{R}),$$

  where $\mu_t^* = \sum_{x \in \operatorname{supp}\mu_t} \delta_x$
- The Varadhan formula

$$\mathbb{P}\{\mu_t = \nu\} \sim e^{-\frac{d^2_{\mathcal{W}}(\operatorname{Leb}_{[0,1]},\nu)}{2t}}, \quad t \ll 1$$

  holds.

Open problems and further projects

- **Extension to higher dimension:** Particles with gravity potential ($\nabla G(x) = \nabla\left(-\frac{1}{2\pi}\ln|x|\right) = -\frac{1}{2\pi}\frac{x}{|x|^2}$ for $d = 2$)

$$dx_i(t) = \frac{a}{2\pi}\sum_{j=1}^{n} m_j \nabla G(x_i(t) - x_j(t))dt + \frac{1}{\sqrt{m_i}}dw_i(t)$$

- **Uniqueness in law:** It is not known if conditions 1)-4) uniquely determine the family X.
- **Other models in physics:** Particle approach to solving of

$$\partial_t \mu_t = \frac{\beta}{2}\Delta\mu_t + \Gamma(\mu_t) + \nabla\cdot[\sqrt{\sigma(\mu_t)}\dot{W}_t]$$

  and connection with the geometry of the Wasserstein space.
- **Investigation of the constructed models.**
- . . . .

Open problems and further projects

- **Extension to higher dimension:** Particles with gravity potential ($\nabla G(x) = \nabla\left(-\frac{1}{2\pi}\ln|x|\right) = -\frac{1}{2\pi}\frac{x}{|x|^2}$ for $d=2$)

$$dx_i(t) = \frac{a}{2\pi}\sum_{j=1}^{n} m_j \nabla G(x_i(t) - x_j(t))dt + \frac{1}{\sqrt{m_i}}dw_i(t)$$

- **Uniqueness in law:** It is not known if conditions 1)-4) uniquely determine the family X.

1. $X(u,0) = u$
2. $X(u,t) \le X(v,t) \quad u < v$
3. $X(u,\cdot)$ is continuous matringale
4. joint quadratic variation is $d\langle X(u,\cdot), X(v,\cdot)\rangle_t = \frac{\mathbb{I}_{\{X(u,t)=X(v,t)\}}}{m(u,t)}dt$,

   where $\pi(u,s) = \{v: \ X(u,t) = X(v,t)\}, \quad m(u,s) = \mathrm{Leb}\,\pi(u,s)$

Open problems and further projects

- **Extension to higher dimension:** Particles with gravity potential ($\nabla G(x) = \nabla\left(-\frac{1}{2\pi}\ln|x|\right) = -\frac{1}{2\pi}\frac{x}{|x|^2}$ for $d = 2$)

$$dx_i(t) = \frac{a}{2\pi}\sum_{j=1}^{n} m_j \nabla G(x_i(t) - x_j(t))dt + \frac{1}{\sqrt{m_i}}dw_i(t)$$

- **Uniqueness in law:** It is not known if conditions 1)-4) uniquely determine the family X.
- **Other models in physics:** Particle approach to solving of

$$\partial_t \mu_t = \frac{\beta}{2}\Delta\mu_t + \Gamma(\mu_t) + \nabla \cdot [\sqrt{\sigma(\mu_t)}\dot{W}_t]$$

and connection with the geometry of the Wasserstein space.

- **Investigation of the constructed models.**
- . . . .

Open problems and further projects

- **Extension to higher dimension:** Particles with gravity potential $(\nabla G(x) = \nabla\left(-\frac{1}{2\pi}\ln|x|\right) = -\frac{1}{2\pi}\frac{x}{|x|^2}$ for $d = 2)$

$$dx_i(t) = \frac{a}{2\pi}\sum_{j=1}^{n} m_j \nabla G(x_i(t) - x_j(t))dt + \frac{1}{\sqrt{m_i}}dw_i(t)$$

- **Uniqueness in law:** It is not known if conditions 1)-4) uniquely determine the family X.
- **Other models in physics:** Particle approach to solving of

$$\partial_t \mu_t = \frac{\beta}{2}\Delta\mu_t + \Gamma(\mu_t) + \nabla\cdot[\sqrt{\sigma(\mu_t)}\dot{W}_t]$$

  and connection with the geometry of the Wasserstein space.
- **Investigation of the constructed models.**
- . . . .

Thank you for your attention!

1. **A particle approximation**
   - *Konarovskyi,* A system of coalescing heavy diffusion particles on the real line, *Ann. Prob.* (2017)
   - *Konarovskyi,* On asymptotic behavior of the modified Arratia flow, *Electron. J. Probab.* (2017)
   - *Konarovskyi,* Cadlag stability and approximation of reversible coalescing-fragmentating Wasserstein dynamics, *arXiv:1711.03011*
2. **Dirichlet form approach**
   - *Konarovskyi, von Renesse,* Reversible Coalescing-Fragmentating Wasserstein Dynamics on the Real Line, *arXiv:1709.02839*
   - *Konarovskyi, von Renesse,* A colloidal Ornstein-Uhlenbeck process, *in preparation*
3. **LDP and Varadhan formula for the coalescing case**
   - *Konarovskyi, von Renesse,* Modified Massive Arratia flow and Wasserstein diffusion, *Comm. Pure Appl. Math.* (2019)
4. **Dean-Kawasaki equation**
   - *Lehmann, Konarovskyi, von Renesse,* Dean-Kawasaki dynamics: Ill-posedness vs. Triviality, *Electron. Comm. Probab.* (2019)
   - *Konarovskyi, Lehmann, von Renesse,* Dean-Kawasaki dynamics with smooth drift potential, *arXiv:1812.11068*