Coalescing diffusion particles on the real line

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Jena, 2014

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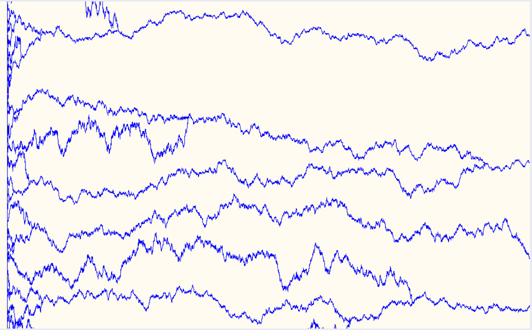
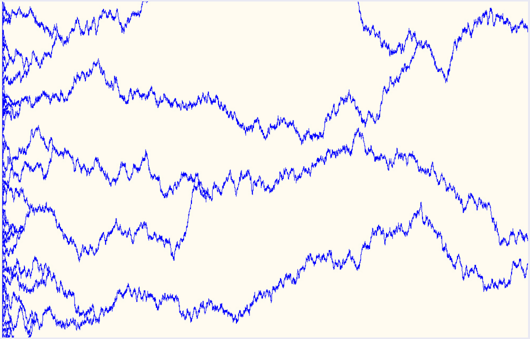
Coalescing diffusion particles with masses



System of diffusion particles on the real line that

1. start from all points of $[0, 1]$;
2. move independently up to the moment of the meeting;
3. coalesce;
4. have mass and diffusion is inversely proportional to mass;
5. masses add after sticking.

Arratia flow

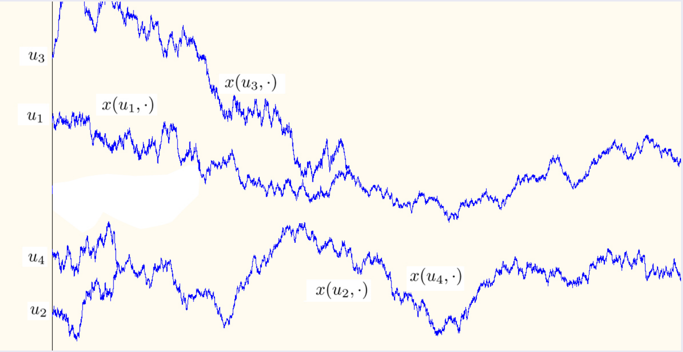


System of diffusion particles on the real line that

1. start from all points of $[0, 1]$;
2. move independently up to the moment of the meeting;
3. coalesce;
4. diffusion is equal to 1;

Existence of Arratia flow

Let $P_{u_1,\ldots,u_n}$ is a distribution of a set of coalescing Brownian particles which start from $u_1, \ldots, u_n$.
$P_{u_1,\ldots,u_n}$, $u_i \in [0,1]$, is a compatible family.



There exists a random process $x(u,\cdot)$, $u \in [0,1]$, with values in $C[0,T]$.

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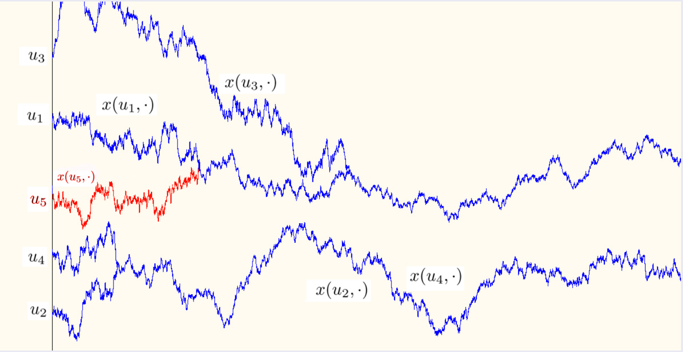

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Modification in Skorochod space $D([0,1], C[0,T])$

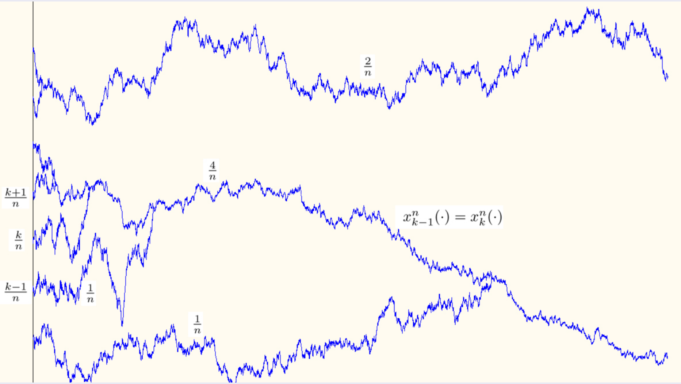
$D([0,1], C[0,T])$ is a space of right continuous functions from $[0,1]$ to $C[0,T]$ with left limits.

Lemma 1

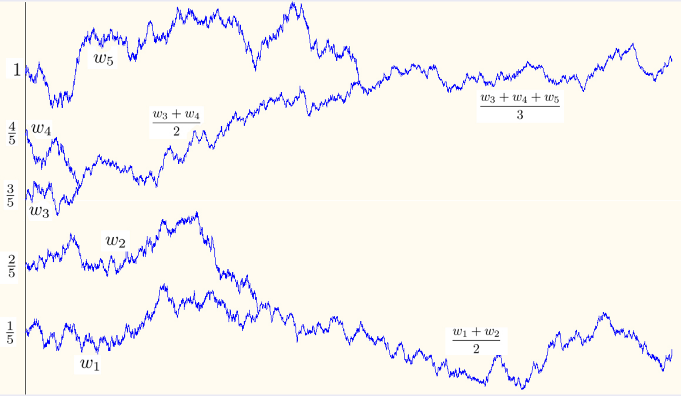
The process $x(u,\cdot),\ u \in [0,1]$, has a modification from $D([0,1], C[0,T])$.

Finite system of particles with masses

Let $x_k^n(t)$, $t \in [0,T]$, $k = 1,\dots,n$, be a system of particles starting from $\frac{k}{n}$, $k = 1,\dots,n$, with masses $\frac{1}{n}$.



Let $w_k(t)$, $t \in [0,T]$, $k = 1,\dots,n$, be a set of Wiener processes starting from $\frac{k}{n}$, $k = 1,\dots,n$, with the diffusions $\frac{1}{n}$.



Theorem 2

The processes $x_k^n(t),\ t \geq 0,\ k = 1, \dots, n$ satisfy the following conditions

1°) $x_k^n(\cdot)$ is a continuous square integrable martingale with respect to the filtration

$$\mathcal{F}_t^n = \sigma(x_i^n(s),\ s \leq t,\ i = 1, \dots, n);$$

2°) $x_k^n(0) = \frac{k}{n},\ k = 1, \dots, n;$

3°) $x_k^n(t) \leq x_l^n(t),\ k < l,\ t \in [0, T];$

4°) $\langle x_k^n(\cdot)\rangle_t = \int\limits_0^t \frac{1}{m_k^n(s)} ds$

where $m_k^n(t) = \frac{1}{n} \#\{i :\ \exists s \leq t\ x_i^n(s) = x_k^n(s)\};$

5°) $\langle x_k^n(\cdot), x_l^n(\cdot)\rangle_t = 0, \quad t < \tau_{k,l}^n,$

where $\tau_{k,l} = \inf\{t :\ x_k^n(t) = x_l^n(t)\}.$

Conditions 1°)–5°) uniquely determine the distribution on $(C[0, T])^n$.

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Lemma 3

There exists constant C, which doesn't depend on n, such that for all k and $t \in (0, T]$

$$\mathbb{E}\frac{1}{m_k^n(t)} \leq \frac{C}{\sqrt{t}}.$$

The proof of lemma follows from inequalities

$$\mathbb{P}\left\{\frac{1}{m_k^n(t)} > r\right\} = \mathbb{P}\left\{m_k^n(t) < \frac{1}{r}, x_l^n(t) - x_k^n(t) > 0\right\} \leq$$
$$\leq \mathbb{P}\left\{\min_{[0,t]}\left(\frac{1}{r} + \sqrt{r}w(s)\right) > 0\right\} = \mathbb{P}\left\{\max_{[0,1]} w(s) < \frac{1}{tr\sqrt{r}}\right\} \leq \frac{C_1}{\sqrt{t}r^{\frac{3}{2}}}$$

and equality

$$\mathbb{E}\frac{1}{m_k^n(t)} = \int_0^\infty \mathbb{P}\left\{\frac{1}{m_k^n(t)} > r\right\} dr.$$

.

Corollary 4

There exists constant C, which doesn't depend on n, such that for all k and $t \in [0, T]$

$$\mathbb{E}\langle x_k^n(\cdot)\rangle_t = \mathbb{E}\int\limits_0^t \frac{1}{m_k^n(s)}ds \leq C\sqrt{t}.$$

This corollary implies that

$$\mathbb{E}\left(x_k^n(t) - \frac{k}{n}\right)^2 \leq C\sqrt{t}.$$

Denote

$$y_n(u,\cdot) = \begin{cases} x_k^n(\cdot), & \frac{k-1}{n} \le u < \frac{k}{n}, \quad k = 1, \dots, n, \\ x_n^n(\cdot), & u = 1. \end{cases}$$

We will interpret $\{y_n(u,\cdot),\ u \in [0,1]\}$ as a random process with values in $C[0,T]$. y_n is a random element in $D([0,1], C[0,T])$.

Lemma 5

For all $\varepsilon > 0$ and u the sequence $\{y_n(u,t),\ t \in [\varepsilon, T]\}_{n \ge 1}$ is tight in $C[\varepsilon, T]$.

Lemma 6

For all $n \in \mathbb{N}$, $u \in [0,2]$, $h \in [0,u]$ and $\lambda > 0$

$$\mathbb{P}\{\|y_n(u+h,\cdot) - y_n(u,\cdot)\| > \lambda,\ \|y_n(u,\cdot) - y_n(u-h,\cdot)\| > \lambda\} \le \frac{Ch^2}{\lambda^2}.$$

Here $y_n(u,\cdot) = y_n(1,\cdot)$, $u \in [1,2]$.

Tightness in $D([0,1], C(0,T])$.

Lemma 7

For all $\varepsilon > 0$ $\{y_n(u,t),\ u \in [0,1],\ t \in [\varepsilon, T]\}$ is tight in $D([0,1], C[\varepsilon, T])$

See Theorem 3.8.6, Theorem 3.8.8 and Corollary 3.8.9.
Ethier S. N. and Kurtz T. G. Markov processes: Characterization and convergence (1996), Wiley, New York.

Corollary 8

$\{y_n(u,t),\ u \in [0,1],\ t \in (0,T]\}$ is tight in $D([0,1], C(0,T])$.

There exists a subsequence $\{n'\}$ such that

$$\{y_{n'}(u,t),\ u \in [0,1],\ t \in (0,T]\} \to \{y(u,t),\ u \in [0,1],\ t \in (0,T]\} \quad \text{in distribution.}$$

So, $\{y(u,t),\ u \in [0,1],\ t \in (0,T]\}$ is a random element in $D([0,1], C(0,T])$.

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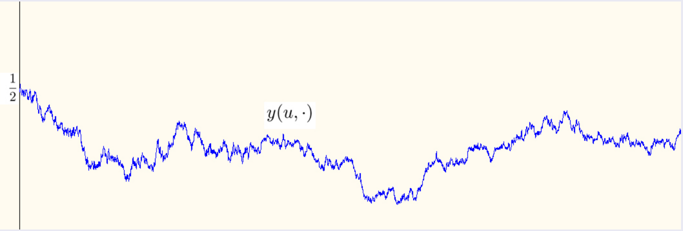
So, $\{y(u,t),\ u \in [0,1],\ t \in (0,T]\}$ is a random element in $D([0,1], C(0,T])$.

Next, we will be interested what properties the process y satisfy, especially

$$y(\cdot, t) \to ?, \quad t \to 0.$$

If particles coalesce very soon then we may have the following.

All particles have coalesced for all $t > 0$



Prolongation on space $D([0,1], C[0,T])$

Lemma 9

Let φ be a twice continuously differentiable function that are bounded together with its derivatives and let

$$\xi_n(t) = \int_0^1 \varphi(y_n(u,t))du, \quad t \in [0,T].$$

Then the sequences $\{\xi_n(t),\ t \in [0,T]\}$ is tight in $C[0,T]$.

Proof. Let us use Aldous' tightness criterion.
Take $\varepsilon > 0$, a set of stoping times $\{\sigma_n\}_{n\geq 1}$ on $[0,T]$ and sequence $\delta_n \searrow 0$.

$$\xi_n(\sigma_n + t) - \xi_n(\sigma_n) = \frac{1}{n}\sum_{k=1}^{n}[\varphi(x_k^n(\sigma_n + t)) - \varphi(x_k^n(\sigma_n))] =$$

$$\frac{1}{n}\sum_{k=1}^{n}\int_0^t \dot{\varphi}(x_k^n(\sigma_n + s))dx_k^n(\sigma_n + s) + \frac{1}{2n}\sum_{k=1}^{n}\int_0^t \frac{\ddot{\varphi}(x_k^n(\sigma_n + s))}{m_k^n(\sigma_n + s)}ds = M(t) + \frac{1}{2}A(t)$$

Estimate $\mathbb{E}|A(t)|$ and $\mathbb{E}|M(t)|$

Define the number of different points $x_k^n(\sigma_n+t)$, $k=1,\ldots,n$ by $\chi_n(t)$. So,

$$\mathbb{E}|A(t)| \le \|\ddot{\varphi}\|\mathbb{E}\int_0^t \chi_n(s)ds \le \|\ddot{\varphi}\|\sum_{k=1}^n \mathbb{E}\gamma_k^n(t),$$

where $\gamma_1^n(t)=t$,

$$\gamma_k^n(t)=\inf\{s:\ x_k^n(\sigma_n+s)=x_{k-1}^n(\sigma_n+s)\}\wedge t,\quad k=2,\ldots,n,$$

are times of free runs of particles x^n on $[\sigma_n,\sigma_n+t]$.
Consider

$$\mathbb{E}\gamma_k^n(t)=\mathbb{E}(\mathbb{E}(\gamma_k^n(t)|\mathcal{F}_{\sigma_n}^{x^n}))=\mathbb{E}(\mathbb{E}_{x^n(\sigma_n)}\widetilde{\gamma}_k^n(t))\le\mathbb{E}(\mathbb{E}_{x^n(\sigma_n)}\widehat{\gamma}_k^n(t))$$

Here $\widetilde{\gamma}^n(t)$ and $\widehat{\gamma}^n(t)$ are times of free runs on $[0,t]$ of particles with mass and of coalescing Brownian particles respectively. So,

$$\mathbb{E}|A(t)| \le \|\ddot{\varphi}\|\sum_{k=1}^n \mathbb{E}(\mathbb{E}_{x^n(\sigma_n)}\widehat{\gamma}_k^n(t)) \le C\mathbb{E}(x_n^n(\sigma_n)-x_1^n(\sigma_n))\sqrt{t}\le C\sqrt{t}.$$

Similarly

$$(\mathbb{E}|M(t)|)^2 \le \mathbb{E}M^2(t) \le \mathbb{E}\left(\frac{1}{n}\sum_{k=1}^{n}\int_0^t \dot{\varphi}(x_k^n(\sigma_n+s))dx_k^n(\sigma_n+s)\right)^2 \le$$

$$\le \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\int_0^t \dot{\varphi}^2(x_k^n(\sigma_n+s))ds \le \|\dot{\varphi}^2\|t.$$

Using the previous estimations and Chebyshev's inequality we obtain

$$\lim_{n\to\infty}\mathbb{P}\{|\xi_n(\sigma_n+\delta_n)-\xi_n(\sigma_n)| \ge \varepsilon\} = 0.$$

Proposition 10

Let φ be a twice continuously differentiable function that are bounded together with its derivatives. Then

$$\int_0^1 \varphi(y(u,t))du \to \int_0^1 \varphi(u)du \quad \text{in probability,} \quad \text{as } t \to 0.$$

Proof. Take a subsequence $\{n'\}$ such that

$$\int_0^1 \varphi(y_{n'}(u,\cdot))du \xrightarrow{d} \xi(\cdot) \quad \text{in } C[0,T].$$

Since the map $F_\varphi : \mathrm{D}([0,1],\mathbb{R}) \to \mathbb{R}$ $F_\varphi(g) = \int_0^1 \varphi(g(u))du$ is continuous,

$$\int_0^1 \varphi(y(u,t))du \stackrel{d}{=} \xi(t) \xrightarrow{a.s.} \xi(0) = \int_0^1 \varphi(u)du.$$

Prolongation on space $D([0,1], C[0,T])$

$$y_n(u,\cdot) \to y(u,\cdot) \quad \text{in} \quad C(0,T] \quad ?$$

Lemma 11

For all $u \in [0,1]$

$$\mathbb{P}\{y(u,\cdot) \neq y(u-,\cdot)\} = 0.$$

Corollary 12

From Lemma 11 we can conclude that for all $u \in [0,1]$

$$y_n(u,\cdot) \to y(u,\cdot) \quad \text{in} \quad C(0,T] \quad \text{in distribution}$$

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Proposition 13

Set

$$y(u,0)=u,\quad u\in[0,1].$$

Then $\{y(u,t),\ u\in[0,1],\ t\in[0,T]\}$ belongs $D([0,1],C[0,T])$ a.s. Furthermore, for all $u\in[0,1]$ $y(u,\cdot)$ is a square integrable $(\mathcal{F}_t)$-martingale, where

$$\mathcal{F}_t=\sigma(y(u,s),\ u\in[0,1],\ s\leq t).$$

Next, we want $\{y(u,t),\ u\in[0,1],\ t\in[0,T]\}$ to describe the evolution of particles starting from all points of $[0,1]$.

Properties

Particles

1. are diffusion;
2. start from all points;
3. coalesce;
4. change mass and diffusion;
5. move independently up to the moment of the meeting.

Theorem 14

$\{y(u,t),\ u\in[0,1],\ t\in[0,T]\}$ satisfies

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$$\mathcal{F}_t=\sigma(y(u,s),\ u\in[0,1],\ s\le t);$$
2. $y(u,0)=u$;
3. $y(u,t)\le y(v,t)$ if $u<v$;
4. $\langle y(u,\cdot)\rangle_t=\int\limits_0^t\frac{ds}{m(u,s)}$,
where $m(u,t)=\lambda\{v:\ \exists s\le t\ y(v,s)=y(u,s)\}$
5. $\langle y(u,\cdot),y(v,\cdot)\rangle_{t\wedge\tau_{u,v}}=0$,
where $\tau_{u,v}=\inf\{t:\ y(u,t)=y(v,t)\}\wedge T$.

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   where $\tau_{u,v} = \inf\{t : \ y(u,t) = y(v,t)\} \wedge T$.

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Theorem 15

For every bounded continuous function φ there exists the limit

$$\int_0^1 \int_0^{\cdot} \varphi(y(u,s))dy(u,s)du := \lim_{\lambda\to 0} \sum_{k=1}^{n} \int_0^{\cdot} \varphi(y(\widetilde{u}_k,s))dy(\widetilde{u}_k,s)\Delta u_k$$

in a space of continuous square integrable martingale and the quadratic characteristic of the limit point is

$$\int_0^{\cdot} \int_0^1 \varphi^2(y(u,s))duds.$$

Moreover, for each twice continuously differentiable function φ that are bounded together with its derivatives we have

$$\int_0^1 \varphi(y(u,t))du = \int_0^1 \varphi(y(u,0))du + \int_0^1 \int_0^t \dot{\varphi}(y(u,s))dy(u,s)du + \frac{1}{2}\int_0^t \int_0^1 \frac{\ddot{\varphi}(y(u,s))}{m(u,s)}duds.$$

Open problems

1. Uniqueness of distribution;
2. Asymptotic growth of the particle mass for small t;
3. Asymptotic behaviour of the particle for small t;

Thank you!