COALESCING-FRAGMENTATING WASSERSTEIN DYNAMICS: PARTICLE APPROACH

Vitalii Konarovskyi*

Universität Leipzig

We construct a family of semimartingales which describe the behavior of a particle system with sticky-reflecting interaction. The model is a physical improvement of the Howitt-Warren flow [20], an infinite system of diffusion particles on the real line which sticky-reflect from each other. But now particles have masses obeying the conservation law and diffusion rate of each particle depends on its mass. The equation which describes the evolution of the particle system is a new type of equations in infinite dimensional space and can be interpreted as an infinite dimensional analog of the equation for sticky-reflected Brownian motion. The particle model appears as a particular solution to the corrected version of the Dean-Kawasaki equation.

1. Introduction. In [31], the author together with von Renesse proposed a class of measure-valued processes, so-called *reversible Coalescing-Fragmentating Wasserstein Dynamics* or shortly *reversible CFWD*, which describes the evolution of mass of particles that interact via some sticky-reflecting mechanism. The construction was aimed at the generalisation of a Brownian motion of a single point (atom) to the case of infinite points (measures) on the real line. The main requirement of such a construction was that the process μ_t had to be reversible in time and its short time asymptotics had to be covered by the Varadhan formula of the form

$$\mathbb{P}\left\{\mu_{t+\varepsilon} = \nu\right\} \sim e^{-\frac{d^2_{\mathcal{W}}(\mu_t,\nu)}{2\varepsilon}}, \quad \varepsilon \ll 1,$$

where $d_{\mathcal{W}}$ denotes the usual Wasserstein distance on the space of probability measures $\mathcal{P}_2(\mathbb{R})$ on the real line with a finite second moment. This led to a new family of measure-valued processes which are naturally connected with

*The author is grateful to Max von Renesse for useful discussions and suggestions. The research was supported by Alexander von Humboldt Foundation

MSC 2010 subject classifications: Primary 60K35, 60B12; secondary 60J60, 60G44, 82B21

Keywords and phrases: Wasserstein diffusion, modified Arratia flow, Howitt-Warren flow, sticky-reflected Brownian motion, infinite dimensional SPDE with discontinuous coefficients

the Riemannian structure of the Wasserstein space of probabilities measures and also to a new class of associated invariant measures for those processes.

The reversible CFWD also solves the corrected Dean-Kawasaki equation[1]

$$d\mu_t = \Delta\mu_t^* dt + \operatorname{div}(\sqrt{\mu_t} dW_t) \tag{1.1}$$

on $\mathcal{P}_2(\mathbb{R})$, where $\mu_t^* = \sum_{x \in \operatorname{supp} \mu_t} \delta_x$ and dW is a white noise vector field on $\mathbb{R}$. It is known that the modified Arratia flow satisfies the same equation (see [33]). This in particular implies the non-uniqueness of solutions to (1.1).

The construction in [31] was based on the Dirichlet form approach. There we proposed a new family of measures on the space $\mathcal{P}_2(\mathbb{R})$ which depends on the interaction potential between particles and then proved an integration by parts formula. This allowed to introduce the naturally associated Dirichlet form $\mathcal{E}$ and construct the corresponding measure-valued process μ_t (a family of processes which depend on the interacting potential between particles). In spite of the power of the Dirichlet form method, such a description has many shortcomings which make the model very complicated for further investigation. In particular,

- the process μ_t, $0 \leq t < \tau$, was defined up to the life time τ and it is unclear in general if the process globally exists, i.e. if τ is infinite a.s.;
- μ_t was defined only for initial distributions μ_0 outside an unknown $\mathcal{E}$-exceptional set;
- although the process describes the evolution of mass of interaction particles, one can say nothing about the behaviour of individual particles;
- the construction does not covers the coalescing interaction between particles that can be considered as a critical case of sticky-reflecting behavior.

The present paper is aimed at the elimination of those defects. For this, we choose a completely different construction. We will approximate an infinite particle system by a finite number of particles. This allows us to construct a continuum collection of ordered continuous semimartingales on the real line which satisfy some natural properties. We also note that the obtained system can be considered as a physical improvement of the Howitt-Warren flow [20, 43] which describes the family of Brownian motions with sticky reflected interaction. The inclusion of the particle mass into the system which influences their motion makes our model much more interesting and natural from the physical point of view.

[1]The Dean-Kawasaki equation is a prototype of equations appearing in fluctuating hydrodynamic theory and has a broad application in the physics (see e.g. [2, 8, 9, 10, 11, 12, 18, 25, 26, 36, 37, 39, 40]). In [29, 30], we showed that the original Dean-Kawasaki equation has either trivial solutions or is ill-posed.

1.1. *Description of the model and formulation of the main results.* We consider a family of diffusion particles on the real line which intuitively can be described as follows. Particles start from some set of points and move keeping the order. Each particle has a mass and fluctuates as a Brownian motion with diffusion rate inversely proportional to their mass. Particles move independently up to the moment of collision with other particles. When some particles collide their masses are added and they form a cluster (a set of particles occupying the same position) that fluctuates also as a Brownian particle with the corresponding diffusion rate. Each particle in a cluster immediately experiences a drift force defined by some interaction potential which makes it leave the cluster.

Let us assume that the total mass of the system is finite. This assumption is needed to overcome some additional difficulties which can occur considering systems of infinite total muss. Moreover, we will for simplicity assume that the total mass equals one. The case of any finite total muss of the system can be obtained by the rescaling of the considered model. Next we describe the dynamics more precisely. Let every particle in the system be labeled by points $u \in (0,1)$ and its position at time $t \geq 0$ be denoted by $X(u,t)$. Since particles keep their order, we assume that $X(u,t) \leq X(v,t)$ for all $u < v$ and t. Each particle u has a mass $m(u,t)$ at time t that is equal to the length of its cluster

$$\pi(u,t) = \{v \in (0,1) : \quad X(u,t) = X(v,t)\}$$

(the set of particles occupying with particle u the same position). According to our requirements, for every u the process $X(u,\cdot)$ has to be a continuous semimartingale with the quadratic variation whose derivative equals $\frac{1}{m(u,t)}$ at time t, that is,

$$d\left[X(u,\cdot)\right]_t = \frac{dt}{m(u,t)}.$$

Since we have assumed that particles move independently up to their collision, it would be reasonable to require that $X(u,t)$ and $X(v,t)$ are independent up to meeting. The problem is that the processes always depend on each other via the mass. So, we replace the condition of independence by zero covariance[2]

$$d\left[X(u,\cdot), X(v,\cdot)\right]_t = 0, \quad \text{provided} \quad X(u,t) \neq X(v,t).$$

In order to define the splitting between the particles, we prescribe a number $\xi(u)$ to each particle u, where ξ is non-decreasing function. This number is

[2]If particles would not change their diffusion rate then this condition would be equivalent to the independent motion of particles at the time when they occupy distinct positions

called an *interaction potential* of particle u. Then particle u, which belongs to a cluster $\pi(u,t)$ at time t, has the drift force

$$\xi(u) - \frac{1}{m(u,t)} \int_{\pi(u,t)} \xi(v)dv$$

that is the difference between own potential and the average potential over the cluster. Summarizing the assumptions above, the family of process $X(u,\cdot)$, $u \in (0,1)$, formally has to solve the following system of equations

$$\begin{aligned} dX(u,t) &= \frac{1}{m(u,t)} \int_{\pi(u,t)} W(dv,dt) \\ &\quad + \left(\xi(u) - \frac{1}{m(u,t)} \int_{\pi(u,t)} \xi(v)dv \right) dt, \end{aligned} \tag{1.2}$$

$u \in (0,1)$, under the restriction $X(u,t) \le X(v,t)$, $u < v$, $t \ge 0$, where W is a Brownian sheet. We also provide (1.2) with the initial condition $X(u,0) = g(u)$.

Let $D([a,b],E)$ denote the Skorohod space of càdlág functions from $[a,b]$ to a Polish space E with the usual Skorohod topology. We say that a function $f : [0,1] \to \mathbb{R}$ is piecewise γ-Hölder continuous if there exists an ordered partition $U = \{u_i,\ i = 1,\ldots,l\}$ of $[0,1]$ such that f is γ-Hölder continuous on each interval (u_{i-1}, u_i), $i \in [l] := \{1,\ldots,l\}$. The first main result of the present paper reads as follows.

THEOREM 1.1. *Let $g, \xi \in D([0,1],\mathbb{R})$ be non-decreasing piecewise $\frac{1}{2}$+-Hölder continuous*[3] *functions on $[0,1]$. Then there exists a random element $X = \{X(u,t),\ t \ge 0,\ u \in [0,1]\}$ in $D([0,1], C([0,\infty)))$ such that*

($R1$) *for all $u \in [0,1]$, $X(u,0) = g(u)$;*
($R2$) *for each $u < v$ from $[0,1]$ and $t \ge 0$, $X(u,t) \le X(v,t)$;*
($R3$) *the process*

$$M^X(u,t) := X(u,t) - g(u) - \int_0^t \left(\xi(u) - \frac{1}{m_X(u,s)} \int_{\pi_X(u,s)} \xi(v)dv \right) ds$$

is a continuous square integrable $(\mathcal{F}_t^X)$-martingale for all $u \in (0,1)$, where $(\mathcal{F}_t^X)$ is the natural filtration generated[4] *by X, $\pi_X(u,t) := \{v : X(u,t) = X(v,t)\}$ and $m_X(u,t) = \mathrm{Leb}\, \pi_X(u,t)$;*

[3] Hereafter we mean that there exists $\varepsilon > 0$ such that the function is $(\frac{1}{2} + \varepsilon)$-Hölder continuous

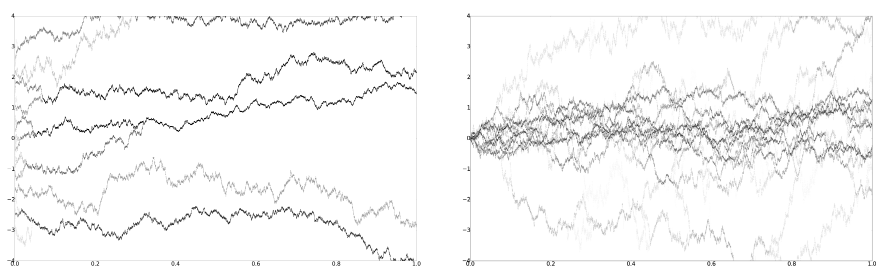
[4] see Section 1.3 and Remark 1.3 for the precise definition

$(R4)$ *the joint quadratic variation of* $M^X(u,\cdot)$ *and* $M^X(v,\cdot)$ *equals*

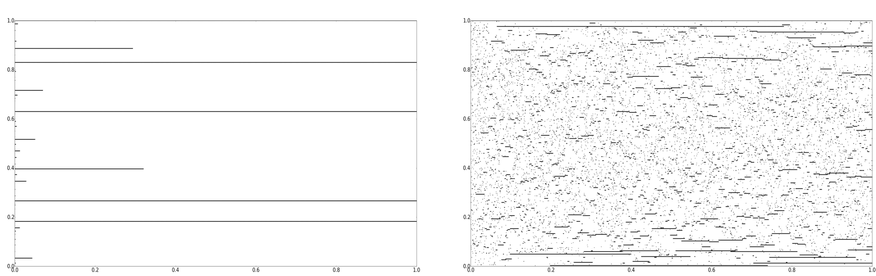
$$[M^X(u,\cdot), M^X(v,\cdot)]_t = \int_0^t \frac{\mathbb{I}_{\{X(u,s)=X(v,s)\}}}{m_X(u,s)} ds.$$

We remark that the random element X from Theorem 1.1 can be interpreted as a weak solution to the system of equations (1.2). In particular, for the coalescing particle system (if $\xi = 0$), Marx in [38] showed that for any family of processes X which satisfies $(R1) - (R4)$ there exists a Brownian sheet W (possibly on an extended probability space) such that X solves system of equations (1.2). We believe that the same result can be obtained for any interaction potential ξ, using the same argument.

We would like to compare the model with the modified Arratia flow of a system of martingales on the real line which satisfies the same conditions with $\xi = 0$ [27, 28, 33], see also [3, 4, 13, 14, 17, 34, 42, 43] for the classical Arratia flow, where particles do not change their diffusion rate. The main difference is an additional drift potential which leads do the dispersion of particles and makes the model very complicated for construction. Moreover, methods proposed there cannot be applied to the sticky-reflected particle system. On the pictures, a computer simulation of both systems is given.



The modified Arratia flow (left) and the sticky-reflected particle system (right) with interacting potential ξ which equals the identity function. Grayscale colour coding is illustrating the atom sizes.



Corresponding clusters behaviour, where dots represent the ends of clusters. Here, two labels u and v belong to the same cluster at time t provided $X(u,t) = X(v,t)$.

In order to construct the family of processes X, we use the approximation of the model by finite particle systems. We first state some estimates for evolution of particle masses in Section 2. It allows to prove the tightness. The main problem is to check that the limiting system of processes satisfies properties $(R1) - (R4)$. To show this, we replace system of equations (1.2) by an equation in some Hilbert space which has discontinuous coefficients and prove that the new equation has solutions. After that we show the connection between solutions to the new equation and system (1.2).

For $p \in [1,\infty]$ let $L_p^{\uparrow}$ denote the space of non-decreasing p-integrable (with respect to the Lebesgue measure on $[0,1]$ denoted by Leb) functions from $[0,1]$ to $\mathbb{R}$, and pr_f be the projection in $L_2 := L_2([0,1], \mathrm{Leb})$ on the linear subspace $L_2(f)$ of $\sigma(f)$-measurable functions. Let also W_t, $t \geq 0$, be a cylindrical Wiener process on L_2. System of equations (1.2) can be rewritten as one SDE in the space $L_2^{\uparrow}$

$$dX_t = \mathrm{pr}_{X_t} dW_t + (\xi - \mathrm{pr}_{X_t} \xi)dt, \quad X_0 = g \tag{1.3}$$

due to the form of the projection operator, where $X_t = X(\cdot, t) \in L_2^{\uparrow}$. The second contribution of the present paper is the development of new methods for solving of equation (1.3), and is establishing of a connection between solutions to such an equation and families of semimartingales satisfying $(R1) - (R4)$. We remark that equation (1.3) can be interpreted as an infinite dimensional analog of the equation for a sticky-reflected Brownian motion on the half line

$$dx(t) = \mathbb{I}_{\{x(t)>0\}} dw(t) + \lambda \mathbb{I}_{\{x(t)=0\}} dt$$

for which the question of existence and uniqueness of solutions is non-trivial (see e.g. [15]). In our case, the uniqueness of solutions to (1.3) remains an open problem.

THEOREM 1.2. *(i) For each $\delta > 0$, $g \in L_{2+\delta}^{\uparrow}$ and $\xi \in L_\infty^{\uparrow}$ there exists a weak solution*[5] *to SDE (1.3).*

(ii) Let $Y = \{Y(u,t),\ u \in [0,1],\ t \geq 0\}$ be a random element in the Skorohod space $D([0,1], C([0,\infty)))$ and X_t, $t \geq 0$, be a continuous process in $L_2^{\uparrow}$ such that $X_t = Y(\cdot, t)$ in L_2 a.s. for all $t \geq 0$. Then the family Y satisfies $(R1) - (R4)$ if and only if the process X_t, $t \geq 0$, is a weak solution to (1.3).

[5] see Definition 1.1

Theorem 1.1 will immediately follow from Theorem 1.2 and the existence of a solution to (1.3) with a modification from the Skorohod space $D([0,1], C([0,\infty)))$ (see the proofs in Section 5).

Next we briefly describe the main idea of proof of Theorem 1.2. The first part of the theorem is proved using a finite particle approximation. We first construct a solution to equation (1.3) if ξ and g are step functions, using the Dirichlet form approach. This corresponds to the case of a finite particle system. Then we approximate any ξ and g by step functions and show that solutions to (1.3) are tight and every limiting process solves equation (1.3). The tightness argument is based on the control of the particle mass, and is rather standard. We recall that, in the case of the modified Arratia flow (if $\xi = 0$), the tightness follows from the estimate

$$\mathbb{P}\{m(u,t) < r\} \leq \frac{C\sqrt{r}}{\sqrt{t}}(g(u+r) - g(u))$$

[28, Lemma 4.1], which can be proved using the coalescing of particles. Now, particles do not coalesce. But we can control the integral $\int_0^t \mathbb{P}\{m(u,s) < r\} ds$ (see lemmas 2.2, 2.4 and 2.5). This is enough for the tightness in Section 3.

A very complicated problem is to check that a limiting process satisfies SDE (1.3). For the modified Arratia flow we showed this, using the fact that a number of distinct particles at each positive time is finite and decreases as time increases because particle coalesce (see Theorem 5.5 [28]). In the sticky-reflected case of interaction, one can prove that the system contains an infinitely many of distinct particles. Namely, if ξ is strictly increasing, then the random set of times at which the particle system consists of an infinite number of distinct particles is tight in $[0,\infty)$ (see our forthcoming paper [32]). So, we cannot use the methods which works for the modified Arratia flow.

Let us roughly explain a new approach which we propose in order to show that a limiting process solves (1.3). Let X^n, $n \geq 1$, solve (1.3) with initial conditions g_n and interacting potentials ξ_n. Let also $\{X^n,\ n \geq 1\}$ converge to X and their quadratic variations $\left\{\int_0^\cdot \mathrm{pr}_{X^n_s} ds,\ n \geq 1\right\}$ to $\int_0^\cdot P_s ds$. For the identification of the limit, it is needed to prove that $P_s = \mathrm{pr}_{X_s}$ for almost all s. Since X^n, $n \geq 1$, are continuous semimartingales, X also is a continuous semimartingale with quadratic variation $\int_0^\cdot P_s ds$. In order to show that $P_s = \mathrm{pr}_{X_s}$, we use the following trick. By the lower semi-continuity of the map $g \mapsto \|\mathrm{pr}_g h\|_{L_2}$ (see Lemma A.4) and the fact that $\mathrm{pr}_{X^n_t}$ is a projection, it is possible to show that P_t is also a projection but maybe on a larger space than $L_2(X_t)$. Then, we prove in Proposition A.3, that the quadratic variation $\int_0^\cdot L_s L_s^* ds$ of any continuous semimartingale Z_t, $t \geq 0$, taking values in $L_2^{\uparrow}$ always satisfies the property $L_t \circ \mathrm{pr}_{Z_t} = L_t$ for almost all t. This immediately implies $P_t = P_t \circ \mathrm{pr}_{X_t} = \mathrm{pr}_{X_t}$. The proposed method will also work for a wider class of SDE on $L_2^{\uparrow}$ with discontinuous coefficients. Proposition A.3 seems to be of independent interest. We also believe that the developed approach is a powerful tool to show that the model can naturally appear as a scaling limit of a discrete interaction particle system. But this question is not considered here.

1.2. *Organisation of the paper.* A general construction of the reversible CFWD via the Dirichlet form approach is recalled in Subsection 2.1. Also a random element in the Skorohod space $D([0,1], C([0,\infty)))$ which satisfies $(R1)-(R4)$ and describes the evolution of a finite particle system is constructed there. The main estimates needed for the tightness are obtained in Subsection 2.2, using properties $(R1)-(R4)$. The core result of Section 3 is the tightness of solutions to (1.3). We also prove there the existence of a corresponding modification in the Skorohod space, by the tightness argument in $D([0,1], C([0,\infty)))$. In Section 4, we show that any limit point solves SDE (1.3), using a purely deterministic result obtained in the appendix (see Subsection A.3). We also prove that its càdlág modification (if it exists) satisfies $(R1)-(R4)$. The construction of a weak solution to (1.3) is done in Section 5. Many auxiliary statements are given in the appendix. In particular, the lower semi-continuity of $g \mapsto \| \mathrm{pr}_g h\|_{L_2}$ is proved in Subsection A.2. The property of the quadratic variation of $L_2^{\uparrow}$-valued continuous semi-martingales that allows to identify a limit is checked in Subsection A.4.

1.3. *Preliminaries and notation.* We will denote the set of non-decreasing càdlág functions from $(0,1)$ to $\mathbb{R}$ by $D^{\uparrow}$. The set of all step functions from $D^{\uparrow}$ with a finite number of jumps is denoted by $\mathcal{S}^{\uparrow}$. If $g \in D^{\uparrow}$ is bounded, then we set

$$g(0) = \lim_{u \downarrow 0} g(u) \quad \text{and} \quad g(1) = \lim_{u \uparrow 1} g(u).$$

Let $(E, \mathcal{F}, P)$ be a complete probability space and $\mathcal{H} \subset \mathcal{F}$. Then $\sigma^*(\mathcal{H})$ denotes the P-completion of $\sigma(\mathcal{H})$. If $g : E \to \mathbb{R}$ is an $\mathcal{F}$-measurable function, then $\sigma^*(g) := \sigma^*(\{g^{-1}(A) : \ A \in \mathcal{B}(\mathbb{R})\})$, where $\mathcal{B}(F)$ denotes the Borel σ-algebra on a topological space F.

Remark 1.1. *We note that $g_1 = g_2$ P-a.e. implies $\sigma^*(g_1) = \sigma^*(g_2)$.*

For $p \in [1, +\infty]$ we denote the space of p-integrable (essential bounded, if $p = +\infty$) functions (more precisely equivalence classes) from $[0,1]$ to $\mathbb{R}$ by

L_p. The usual norm in L_p is denoted by $\|\cdot\|_{L_p}$ and the usual inner product in L_2 by $(\cdot,\cdot)_{L_2}$.

For a Borel measurable function $g:(0,1)\to\mathbb{R}$ the space of all $\sigma^*(g)$-measurable functions from L_2 is denoted by $L_2(g)$. By Remark 1.1, $L_2(g)$ is well-defined for every equivalence class g from L_p.

Let $\mathcal{L}_2(L_2)$ denote the space of Hilbert-Schmidt operators on L_2 with the inner product given by

$$(A,B)_{HS}=\sum_{i=1}^{\infty}(Ae_i,Be_i)_{L_2},\quad A,B\in\mathcal{L}_2(L_2), \tag{1.4}$$

where $\{e_i,\ i\in\mathbb{N}\}$ is an orthonormal basis of L_2. We note that the inner product does not depend on the choice of basis $\{e_i,\ i\in\mathbb{N}\}$. The corresponding norm in $\mathcal{L}_2(L_2)$ is denoted by $\|\cdot\|_{HS}$.

If H is a Hilbert space with the inner product $(\cdot,\cdot)_H$, then $L_2([0,T],H)$ will denote the Hilbert space of 2-integrable H-valued functions on $[0,T]$ endowed with the inner product

$$(f,g)_{T,H}=\int_0^T(f_t,g_t)_H dt,\quad f,g\in L_2([0,T],H).$$

The corresponding norm is denoted by $\|\cdot\|_{T,H}$. If $H=\mathcal{L}_2(L_2)$, then the inner product and the norm will be denoted by $(\cdot,\cdot)_{T,HS}$ and $\|\cdot\|_{T,HS}$, respectively.

Let $C(I,E)$ denote the space of continuous functions from $I\subset\mathbb{R}$ to a Banach space E equipped with the topology of uniform convergence on compacts. For simplicity we also write $C(I)$ instead of $C(I,\mathbb{R})$. If I is a compact set, then the uniform norm will be denoted by $\|\cdot\|_{C(I,E)}$. In the case $E=\mathbb{R}$, the uniform norm is denoted by $\|\cdot\|_{C(I)}$.

The set of all infinitely differentiable real-valued functions on $\mathbb{R}^m$ with all partial derivatives bounded is denoted by $C_b^\infty(\mathbb{R}^m)$ and $C_0^\infty(\mathbb{R}^m)$ is the set of functions from $C_b^\infty(\mathbb{R}^m)$ with compact support.

Let $D([a,b],E)$ denote the space of càdlág functions from $[a,b]$ to a Polish space E with the usual Skorohod distance (see e.g. Section 3 [5] and Section A.5).

The Lebesgue measure on $\mathbb{R}$ will be denoted by Leb.

The set of functions from L_p which have a non-decreasing modification is denoted by $L_p^\uparrow$. By Proposition A.1 [28], $L_2^\uparrow$ is a closed set in L_2 and each $f\in L_2^\uparrow$ has a unique modification from $D^\uparrow$. So, considering an element from $L_2^\uparrow$ as a function, we will always take its modification from $D^\uparrow$. We also set $L_p^\uparrow(\xi):=L_p^\uparrow\cap L_p(\xi)$.

For $g \in L_2^{\uparrow}$ we denote the projection operator in L_2 on the closed linear subspace $L_2(g)$ by pr_g. Let $\#g$ denote the number of distinct points of the set $\{g(u),\ u \in (0,1)\}$, where the modification g is taken from $D^{\uparrow}$. We will prove in Subsection A.2 (see Lemma A.3 there) that $\#g = \|\mathrm{pr}_g\|_{HS}^2$.

Remark 1.2. *Since pr_g maps $L_2^{\uparrow}$ into $L_2^{\uparrow}$ (see e.g. Lemma A.2 below), for every $\xi \in L_2^{\uparrow}$ and $u \in (0,1)$ we will understand $\left(\mathrm{pr}_g \xi\right)(u)$ as a value of the function $f \in D^{\uparrow}$ at u, where $\mathrm{pr}_g \xi = f$ a.e., and*

$$\left(\mathrm{pr}_g \xi\right)(0) = \lim_{u \downarrow 0} f(u) \quad \text{and} \quad \left(\mathrm{pr}_g \xi\right)(1) = \lim_{u \uparrow 1} f(u),$$

if the limits exist.

We denote the filtration generated by a process X_t, $t \geq 0$, by $(\mathcal{F}_t^{\circ,X})_{t\geq 0}$, that is, $\mathcal{F}_t^{\circ,X} = \sigma(X_t,\ s \leq t)$, $t \geq 0$. The smallest right-continuous and complete extension of $(\mathcal{F}_t^{\circ,X})_{t\geq 0}$ is denoted by $(\mathcal{F}_t^X)_{t\geq 0}$ (see e.g. Lemma 7.8 [24] for existence). The filtration $(\mathcal{F}_t^X)_{t\geq 0}$ is called the *natural filtration* generated by X.

Remark 1.3. *If X_t, $t \geq 0$, is an L_2-valued process and $\{Y(u,t),\ u \in [0,1],\ t \geq 0\}$ is an random element in $D([0,1], C([0,\infty)))$ such that $X_t = Y(\cdot,t)$ in L_2 a.s. for all $t \geq 0$, then $(\mathcal{F}_t^X)_{t\geq 0}$ coincides with the smallest right-continuous and complete extension of the filtration*

$$(\sigma(Y(u,s),\ u \in [0,1],\ s \leq t))_{t\geq 0}\,.$$

This can be proved using e.g. Lemma 4.4 below.

Now we give a definition of weak solution to equation (1.3).

Definition 1.1. *An $L_2^{\uparrow}$-valued random process X_t, $t \geq 0$, is called a weak solution to SDE (1.3) if*

- $(E1)$ *$X_0 = g$;*
- $(E2)$ *$X_t \in C([0,\infty), L_2^{\uparrow})$;*
- $(E3)$ *$\mathbb{E}\|X_t\|_{L_2}^2 < \infty$ for all $t \geq 0$;*
- $(E4)$ *the process*

$$M_t^X := X_t - g - \int_0^t (\xi - \mathrm{pr}_{X_s} \xi) ds, \quad t \geq 0,$$

is a continuous square integrable $(\mathcal{F}^X)$*-martingale*[6] *in* L_2 *with the quadratic variation process*

$$\langle\langle M^X \rangle\rangle_t = \int_0^t \operatorname{pr}_{X_s} ds.$$

Remark 1.4. *(i) The process*

$$A_t^X := \int_0^t (\xi - \operatorname{pr}_{X_s} \xi) ds, \quad t \geq 0,$$

is continuous in L_2.

(ii) Condition $(E4)$ *is equivalent to*

$(E'4)$ *For each* $t \geq 0$ $\mathbb{E}\|M_t^X\|_{L_2}^2 < \infty$ *and for each* $h \in L_2$ *the process*

$$(M_t^X, h)_{L_2} = (X_t, h)_{L_2} - (g, h)_{L_2} - \int_0^t (\xi - \operatorname{pr}_{X_s} \xi, h)_{L_2} ds, \quad t \geq 0,$$

is a continuous square integrable $(\mathcal{F}^X)$*-martingale with the quadratic variation*

$$[(M_\cdot^X, h)_{L_2}]_t = \int_0^t \|\operatorname{pr}_{X_s} h\|_{L_2}^2 ds.$$

(iii) For each $t \geq 0$ $\mathbb{E}\|X_t\|_{L_2}^2 < \infty$ *provided* $\mathbb{E}\|M_t^X\|_{L_2}^2 < \infty$*, since* $\|A_t^X\|_{L_2} \leq 2\|\xi\|_{L_2} t$.

(iv) Similarly as in the proof of Lemma 2.1 [19], one can show that the increasing process of M^X *is given by*

$$\langle M^X \rangle_t = \int_0^t \|\operatorname{pr}_{X_s}\|_{HS}^2 ds, \quad t \geq 0,$$

that is,

$$\|M_t^X\|_{L_2}^2 - \int_0^t \|\operatorname{pr}_{X_s}\|_{HS}^2 ds, \quad t \geq 0,$$

is an $(\mathcal{F}^X)$*-martingale. In particular,* $\mathbb{E}\|M_t^X\|_{L_2}^2 = \mathbb{E}\int_0^t \|\operatorname{pr}_{X_s}\|_{HS}^2 ds < \infty$ *for all* $t \geq 0$.

(v) If X *is a weak solution to SDE (1.3), then there exists a cylindrical Wiener process* W_t, $t \geq 0$, *in* L_2 *(maybe on an extended probability space) such that*

$$X_t = g + \int_0^t \operatorname{pr}_{X_s} dW_s + \int_0^t (\xi - \operatorname{pr}_{X_s} \xi) ds, \quad t \geq 0,$$

by Corollary 2.2 [19].

[6]see Section 2.1.3 [19] for the introduction to martingales in a Hilbert space

2. A finite sticky-reflected particle system.

2.1. *The general construction via the Dirichlet form approach.* In this section, we recall the construction of a weak solution to SDE (1.3) for some class of functions g and ξ, using the Dirichlet form approach. Namely, we are going to construct a reversible CFWD for "almost all" $g \in L_2^{\uparrow}(\xi)$, as in [31]. In the case $\xi \in \mathcal{S}^{\uparrow}$, we also show that the constructed process has a modification from the Skorohod space satisfying $(R1)$-$(R4)$. So, let $\xi \in D^{\uparrow}$ be a fixed bounded function.

We first introduce a measure Ξ^{ξ} on $L_2^{\uparrow}$ which plays a role of an invariant measure for the reversible CFWD X_t, $t \geq 0$. We set for each $n \in \mathbb{N}$

$$E^n := \{x = (x_k)_{k\in[n]} \in \mathbb{R}^n : \ x_1 \leq \ldots \leq x_n\}$$

and

$$Q^n := \{q = (q_k)_{k\in[n-1]} \in [0,1]^{n-1} : \ q_1 < \ldots < q_{n-1}\}, \quad \text{if } \ n \geq 2,$$

where $[n] := \{1, \ldots, n\}$. Considering $q \in Q^n$, we will always take $q_0 = 0$ and $q_n = 1$ for convenience. Let $\chi_1 : \mathbb{R} \to L_2^{\uparrow}$ and $\chi_n : E^n \times Q^n \to L_2^{\uparrow}$, $n \geq 2$, be given by

$$\chi_1(x) := x\mathbb{I}_{[0,1]} \quad \text{and} \quad \chi_n(x,q) := \sum_{k=1}^{n} x_k \mathbb{I}_{[q_{k-1},q_k)} + x_n \mathbb{I}_{\{1\}},$$

where $\mathbb{I}_A$ is the indicator function of a set A. Setting

$$c_n(q) := \prod_{k=1}^{n} (q_k - q_{k-1}), \quad n \geq 2,$$

we define the measure on $L_2^{\uparrow}$ as follows:

$$\Xi^{\xi}(B) := \int_{\mathbb{R}} \mathbb{I}_B(\chi_1(x))dx + \sum_{n=2}^{\infty} \int_{Q^n} \left[c_n(q) \int_{E^n} \mathbb{I}_B(\chi_n(x,q))dx \right] d\xi^{\otimes(n-1)}(q)$$

for all $B \in \mathcal{B}(L_2^{\uparrow})$. Here, $\int_{Q^n} \ldots d\xi^{\otimes(n-1)}(q)$ is the $(n-1)$-dim Lebesgue-Stieltjes integral with respect to $\xi^{\otimes(n-1)}(q) = \xi(q_1) \cdot \ldots \cdot \xi(q_{n-1})$.

The measure Ξ^{ξ} was first proposed in Section 4 [31].

Proposition 2.1. *The measure Ξ^{ξ} is a σ-finite measure on $L_2^{\uparrow}$ with* $\operatorname{supp} \Xi^{\xi} = L_2^{\uparrow}(\xi)$.

PROOF. The proof of the proposition was given in [31]. See Lemma 4.2 (ii), Remark 4.4 and Proposition 4.7 there. □

Next, we denote the linear space generated by functions on $L_2^{\uparrow}$ of the form

$$U = \vartheta\left((\cdot, h_1)_{L_2}, \ldots, (\cdot, h_m)_{L_2}\right)\varphi\left(\|\cdot\|_{L_2}^2\right) = \vartheta\left((\cdot, \mathbf{h})_{L_2}\right)\varphi\left(\|\cdot\|_{L_2}^2\right) \tag{2.1}$$

by $\mathcal{FC}$, where $\vartheta \in C_b^\infty(\mathbb{R}^m)$, $\varphi \in C_0^\infty(\mathbb{R})$ and $h_j \in L_2$, $j \in [m]$.

For each $U \in \mathcal{FC}$ we introduce its derivative as follows

$$\mathrm{D}U(g) := \mathrm{pr}_g\left[\nabla^{L_2} U(g)\right], \quad g \in L_2^{\uparrow},$$

where ∇^{L_2} denotes the Fréchet derivative on L_2. If U is given by (2.1), then a simple calculation shows that

$$\mathrm{D}U(g) = \varphi\left(\|g\|_{L_2}^2\right)\sum_{j=1}^m \partial_j\vartheta\left((g, \mathbf{h})_{L_2}\right)\mathrm{pr}_g\, h_j + 2\vartheta\left((g, \mathbf{h})_{L_2}\right)\varphi'\left(\|g\|_{L_2}^2\right) g \tag{2.2}$$

for all $g \in L_2^{\uparrow}$, where $\partial_j\vartheta(x) := \frac{\partial}{\partial x_j}\vartheta(x)$, $x \in \mathbb{R}^m$.

The following integration by parts formula was proved in [31] (see Theorem 5.6 there).

THEOREM 2.1. *For each $U, V \in \mathcal{FC}$*

$$\begin{aligned}\int_{L_2^{\uparrow}} (\mathrm{D}U(g), \mathrm{D}V(g))_{L_2}\Xi^\xi(dg) &= -\int_{L_2^{\uparrow}} L_0U(g)V(g)\Xi^\xi(dg)\\ &\quad - \int_{L_2^{\uparrow}} V(g)(\nabla^{L_2}U(g), \xi - \mathrm{pr}_g\,\xi)_{L_2}\Xi^\xi(dg),\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}L_0U(g) &= \varphi\left(\|g\|_{L_2}^2\right)\sum_{i,j=1}^m \partial_i\partial_j\vartheta\left((g, \mathbf{h})_{L_2}\right)(\mathrm{pr}_g\, h_i, \mathrm{pr}_g\, h_j)_{L_2}\\ &\quad + \vartheta\left((g, \mathbf{h})_{L_2}\right)\left[4\varphi''\left(\|g\|_{L_2}^2\right)\|g\|_{L_2}^2 + 2\varphi'\left(\|g\|_{L_2}^2\right)\cdot \#g\right]\\ &\quad + 2\sum_{j=1}^m \partial_j\vartheta\left((g, \mathbf{h})_{L_2}\right)\varphi'\left(\|g\|_{L_2}^2\right)(\mathrm{pr}_g\, h_j, g)_{L_2},\end{aligned}$$

if U is defined by (2.1).

REMARK 2.1. *We note that $\#g$ is finite only for $g \in \mathcal{S}^{\uparrow}$. Since $\Xi^\xi(L_2^{\uparrow} \setminus \mathcal{S}^{\uparrow}) = 0$, the function L_0U is well-defined Ξ^ξ-a.e. for all $U \in \mathcal{FC}$. Moreover, it belongs to $L_2(L_2^{\uparrow}, \Xi^\xi)$, by Lemma 4.2 [31].*

Since $\operatorname{supp}\Xi^\xi = L_2^\uparrow(\xi)$, we will define a bilinear form on $L_2(L_2^\uparrow(\xi), \Xi^\xi)$. We set

$$\mathcal{E}^\xi(U,V) = \frac{1}{2}\int_{L_2^\uparrow(\xi)} (\mathrm{D}U(g), \mathrm{D}V(g))_{L_2}\Xi^\xi(dg), \quad U,V \in \mathcal{FC}.$$

Then $(\mathcal{E}^\xi, \mathcal{FC})$ is a densely defined positive definite symmetric bilinear form on $L_2(L_2^\uparrow(\xi), \Xi^\xi)$. Moreover, Theorem 2.1 and Proposition I.3.3 [35] imply that $(\mathcal{E}^\xi, \mathcal{FC})$ is closable on $L_2(L_2^\uparrow(\xi), \Xi^\xi)$. Its closure will be denoted by $(\mathcal{E}^\xi, \mathbb{D}^\xi)$.

THEOREM 2.2. *For each bounded $\xi \in D^\uparrow$ the bilinear form $(\mathcal{E}^\xi, \mathbb{D}^\xi)$ is a quasi-regular local*[7] *symmetric Dirichlet form on $L_2(L_2^\uparrow(\xi), \Xi^\xi)$. Moreover, if ξ is constant on some neighbourhoods of 0 and 1, then $(\mathcal{E}^\xi, \mathbb{D}^\xi)$ is strictly quasi-regular and conservative.*

PROOF. The proof of the theorem can be found in [31]. The fact that $(\mathcal{E}^\xi, \mathbb{D}^\xi)$ is a Dirichlet form, the quasi-regularity and the local property were proved in propositions 5.14, 6.5 and 6.6, respectively. The strictly quasi-regularity and conservativeness were proved in Proposition 6.9. □

By theorems IV.6.4, V.1.11 [35] and Theorem 2.2, there exists a diffusion process[8] $\tilde{X} = \left(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \{\tilde{X}_t\}_{t\geq 0}, \{\tilde{\mathbb{P}}_g\}_{g\in L_2^\uparrow(\xi)_\Delta}\right)$ with state space $L_2^\uparrow(2\xi) = L_2^\uparrow(\xi)$ and life time ζ that is properly associated with $(\mathcal{E}^{2\xi}, \mathbb{D}^{2\xi})$[9]. Furthermore, if ξ is constant on some neighbourhoods of 0 and 1, then $\tilde{X}$ is a Hunt process with the infinite life time.

We set

$$\tilde{M}_t := \tilde{X}_t - \tilde{X}_0 - \int_0^t (\xi - \operatorname{pr}_{\tilde{X}_s}\xi)ds, \quad t \geq 0,$$

and denote the expectation with respect to $\tilde{\mathbb{P}}_g$ by $\tilde{\mathbb{E}}_g$.

PROPOSITION 2.2. *Let ξ is constant on some neighbourhoods of 0 and 1. Then there exists a set $\Theta_\xi \subseteq L_2^\uparrow(\xi)$ with $\mathcal{E}^{2\xi}$-exceptional complement (in $L_2^\uparrow(\xi)$) such that for every $g \in \Theta_\xi$ $\tilde{\mathbb{E}}_g\|\tilde{X}_t\|_2^2 < \infty$, $t \geq 0$, and for each $h \in L_2$ the process*

$$(\tilde{M}_t, h)_{L_2} = (\tilde{X}_t, h)_{L_2} - (\tilde{X}_0, h)_{L_2} - \int_0^t (\xi - \operatorname{pr}_{\tilde{X}_s}\xi, h)_{L_2}ds, \quad t \geq 0,$$

[7]For the definition of quasi-regularity, strictly quasi-regularity and local property see def. IV.3.1, V.2.11 and V.1.1 [35], respectively.

[8]see Definition V.1.10 [35]

[9]We consider the interaction potential 2ξ instead of ξ in order to obtain solutions to SDE with the drift term $(\xi - \operatorname{pr}_{X_t}\xi)dt$ instead of $\frac{1}{2}(\xi - \operatorname{pr}_{X_t}\xi)dt$ (see Section 8 [31]).

is a continuous square integrable $(\tilde{\mathcal{F}}_t)$*-martingale under* $\tilde{\mathbb{P}}_g$ *with the quadratic variation*

$$[(\tilde{M}_\cdot, h)_{L_2}]_t = \int_0^t \|\operatorname{pr}_{\tilde{X}_s} h\|_{L_2}^2 ds, \quad t \geq 0.$$

In particular, X *is a weak solution to SDE* (1.3) *on the probability space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}_g)$.

PROOF. See Corollary 8.2 [31] for the proof of the proposition. □

In the rest of this subsection we suppose that $\xi = \sum_{k=1}^n \varsigma_k \mathbb{I}_{\pi_k} \in \mathcal{S}^\uparrow$, where $\varsigma_k < \varsigma_{k+1}$, $k \in [n-1]$, and $\{\pi_k,\ k \in [n]\}$ is a partition of $[0,1]$.

Let $\tilde{X}(\cdot, t, \omega)$ denote the modification of $\tilde{X}_t(\omega)$ from $D^\uparrow$ for each $\omega \in \tilde{\Omega}$ and $t \geq 0$. Since $\tilde{X}$ takes values in the space $L_2^\uparrow(\xi)$, it is easy to see that

$$\tilde{X}(u,t) = \sum_{k=1}^n \tilde{x}_k(t) \mathbb{I}_{\pi_k}(u), \quad u \in [0,1], \quad t \geq 0,$$

where $\tilde{x}_k(t) = \frac{1}{\operatorname{Leb}(\pi_k)} (\tilde{X}_t, \mathbb{I}_{\pi_k})_{L_2}$, by Proposition A.2 [31]. This yields that the process $\tilde{X}(u,t)$, $t \geq 0$, is continuous for every $u \in [0,1]$.

PROPOSITION 2.3. *The process* $\{\tilde{X}(u,t),\ u \in [0,1],\ t \geq 0\}$ *belongs to the Skorohod space* $D([0,1], C([0,\infty))$ *and for each* $g \in \Theta_\xi$ *it satisfies properties* $(R1) - (R4)$ *on the probability space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}_g)$.

PROOF. The statement follows from Proposition 2.2 and the following property of the projection operator:

$$(\operatorname{pr}_f h^u, \operatorname{pr}_f h^v)_{L_2} = \frac{\mathbb{I}_{\{f(u)=f(v)\}}}{m_f(u)}$$

for all $u, v \in [0,1]$ and $f = \sum_{k=1}^n y_i \mathbb{I}_{\pi_k} \in \mathcal{S}^\uparrow$, where $h^u := \frac{1}{\operatorname{Leb}(\pi_k)} \mathbb{I}_{\pi_k}$ with k satisfying $u \in \pi_k$ and $m_f(u) := \operatorname{Leb}\{v :\ f(u) = f(v)\}$.

We omit the detailed proof, since we will prove Theorem 1.2 (ii) in a more general setting later. □

2.2. *Properties of a finite system.* In this section, we study some properties of random elements from $D([0,1], C([0,\infty))$ satisfying $(R1) - (R4)$. In particular, we obtain some estimates for diffusion rates of individual particles. These properties will be later used in order to prove the tightness in Section 3. Let us note that, in the previous section, we have constructed such elements only for $\mathcal{E}$-q.e. $g \in L_2^\uparrow(\xi)$. By Exercise III.2.3. [35], they are constructed for Ξ^ξ-almost all $g \in L_2^\uparrow(\xi)$. In spite of this, we fix $\xi, g \in \mathcal{S}^\uparrow$ (not necessary $g \in L_2^\uparrow(\xi)$) and suppose that a process $\{X(u,t),\ u \in [0,1],\ t \geq 0\}$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, exists and satisfies $(R1)-(R4)$. We recall that $(\mathcal{F}_t^X)_{t\geq 0}$ coincides with the smallest right-continuous and complete extension of the filtration

$$(\sigma(X(u,s),\ u \in [0,1],\ s \leq t))_{t\geq 0}\,,$$

by Remark 1.3.

For simplicity of notation we set $\mathcal{F}_t := \mathcal{F}_t^X$, $m(u,t) := m_X(u,t)$ and $M(u,t) := M^X(u,t)$, $u \in [0,1]$, $t \geq 0$.

Lemma 2.1. *If $\xi(u) = \xi(v)$ for some $u, v \in [0,1]$, then*

$$\mathbb{P}\left\{X(u,t) = X(v,t) \quad \textit{implies} \quad X(u,t+s) = X(v,t+s) \quad \forall s \geq 0\right\} = 1.$$

Proof. We assume that $u > v$. By $(R2)$, $(R3)$ and Lemma A.2,

$$\begin{aligned} X(u,t) - X(v,t) &= M(u,t) - M(v,t) + g(u) - g(v) \\ &\qquad - \int_0^t \left[\left(\mathrm{pr}_{X_s}\xi\right)(u) - \left(\mathrm{pr}_{X_s}\xi\right)(v)\right] ds, \quad t \geq 0, \end{aligned}$$

is a continuous positive supermartingale, since $\left(\mathrm{pr}_{X_s}\xi\right)(u) - \left(\mathrm{pr}_{X_s}\xi\right)(v) \geq 0$, $s \geq 0$. Thus, the statement follows from Proposition II.3.4 [41]. $\square$

Remark 2.2. *Lemma 2.1 remains also valid for any $\xi, g \in D^\uparrow$ (if X exists).*

Corollary 2.1. *If $\xi(u) = \xi(v)$ and $g(u) = g(v)$ for some $u, v \in [0,1]$, then $X(u,\cdot) = X(v,\cdot)$ a.s. Moreover, there exists a partition $\{\pi_k,\ k \in [n]\}$ of $[0,1]$ and a system of continuous processes $\{x_k(t),\ t \geq 0,\ k \in [n]\}$ such that almost surely*

$$X(u,t) = \sum_{k=1}^n x_k(t)\mathbb{I}_{\pi_k}(u), \quad u \in [0,1], \quad t \geq 0.$$

Proof. The first part of the corollary immediately follows from Lemma 2.1. To prove the second part, we first note that ξ and g (from $\mathcal{S}^\uparrow$) can be written as

$$\xi = \sum_{k=1}^n \varsigma_k \mathbb{I}_{\pi_k} \quad \text{and} \quad g = \sum_{k=1}^n y_k \mathbb{I}_{\pi_k},$$

for some partition $\{\pi_k,\ k \in [n]\}$ of $[0,1]$, $\varsigma_k \leq \varsigma_{k+1}$ and $y_k \leq y_{k+1}$, $k \in [n-1]$. Hence, taking $x_k(\cdot) := X(u_k,\cdot)$, for some $u_k \in \pi_k$, the needed equality follows from the first part of the corollary and $(R2)$. The corollary is proved. $\square$

Corollary 2.2. *For each* $u \in [0,1]$ *there exists a non-random constant* $\delta_u > 0$ *such that* $\inf_{t\geq 0} m(u,t) \geq \delta_u$ *a.s.*

Proof. Taking $\delta_u = \sup\{v_2 - v_1 : \ g(v_1) = g(u) = g(v_2)$ and $\xi(v_1) = \xi(u) = \xi(v_2)\}$, the inequality easily follows from Corollary 2.1. $\square$

Next, we introduce a function which will be used in the following statements:

$$
\begin{aligned}
G(r_1, r_2, u, t) &:= 2(g(u+r_2) - g(u))(g(u) - g(u-r_1)) \\
&\quad + 2(\xi(u) - \xi(u-r_1))\left[t(g(u+r_2) - g(u)) + \frac{t^2}{2}(\xi(u+r_2) - \xi(u))\right] \\
&\quad + 2(\xi(u+r_2) - \xi(u))\left[t(g(u) - g(u-r_1)) + \frac{t^2}{2}(\xi(u) - \xi(u-r_1))\right].
\end{aligned} \tag{2.4}
$$

Lemma 2.2. *For each* $u \in (0,1)$, $0 < r \leq u \wedge (1-u)$ *and* $t \geq 0$ *the inequality*

$$\int_0^t \mathbb{P}\{m(u,s) < r\} ds \leq r G(r,r,u,t)$$

holds.

Proof. We fix u, r as in the assumption of the lemma and denote

$$Z_+(t) := X(u+r,t) - X(u,t), \quad Z_-(t) := X(u,t) - X(u-r,t)$$

for all $t \geq 0$.

Then Z_+ and Z_- can be written as follows

$$
\begin{aligned}
Z_+(t) &= z_+ + N_+(t) + \int_0^t b_+(s) ds, \\
Z_-(t) &= z_- + N_-(t) + \int_0^t b_-(s) ds,
\end{aligned}
$$

for all $t \geq 0$, where

$$
\begin{aligned}
z_+ &:= g(u+r) - g(u), \quad z_- := g(u) - g(u-r), \\
b_+(t) &:= \xi(u+r) - \xi(u) - \left[\left(\mathrm{pr}_{X_t} \xi\right)(u+r) - \left(\mathrm{pr}_{X_t} \xi\right)(u)\right], \\
b_-(t) &:= \xi(u) - \xi(u-r) - \left[\left(\mathrm{pr}_{X_t} \xi\right)(u) - \left(\mathrm{pr}_{X_t} \xi\right)(u-r)\right]
\end{aligned}
$$

and the square integrable martingales N_+, N_- are defined as Z_+ and Z_- with X replaced by M.

Since the projection of a non-decreasing function is also non-decreasing (see Lemma A.2), we have that

$$b_+(t) \le \xi(u+r) - \xi(u) \quad \text{and} \quad b_-(t) \le \xi(u) - \xi(u-r) \tag{2.5}$$

for all $t \ge 0$.

Next, using $(R4)$, we evaluate the joint variation of N_+ and N_-. So, for $t \ge 0$ we have

$$\begin{aligned}
[N_+, N_-]_t &= [M(u+r,\cdot) - M(u,\cdot), M(u,\cdot) - M(u-r,\cdot)]_t \\
&= \int_0^t \left[\frac{\mathbb{I}_{\{Z_+(s)=0\}}}{m(u,s)} + \frac{\mathbb{I}_{\{Z_-(s)=0\}}}{m(u,s)} - \frac{1}{m(u,s)} \right. \\
&\qquad \left. - \frac{\mathbb{I}_{\{X(u+r,s)=X(u-r,s)\}}}{m(u,s)} \right] ds = - \int_0^t \frac{\mathbb{I}_{\{Z_+(s)>0, Z_-(s)>0\}}}{m(u,s)} ds.
\end{aligned}$$

Thus, Itô's formula implies

$$\begin{aligned}
Z_+(t)Z_-(t) &= z_+ z_- + \int_0^t Z_+(s) dN_-(s) + \int_0^t Z_-(s) dN_+(s) \\
&\quad + \int_0^t Z_+(s) b_-(s) ds + \int_0^t Z_-(s) b_+(s) ds - \int_0^t \frac{\mathbb{I}_{\{Z_+(s)>0, Z_-(s)>0\}}}{m(u,s)} ds.
\end{aligned}$$

Taking the expectation, we obtain

$$\begin{aligned}
&\mathbb{E} Z_+(t) Z_-(t) + \mathbb{E} \int_0^t \frac{\mathbb{I}_{\{Z_+(s)>0, Z_-(s)>0\}}}{m(u,s)} ds \\
&\qquad = z_+ z_- + \mathbb{E} \int_0^t Z_+(s) b_-(s) ds + \mathbb{E} \int_0^t Z_-(s) b_+(s) ds.
\end{aligned} \tag{2.6}$$

Next, we estimate the right hand side of the obtained equality, using estimates (2.5). So,

$$\begin{aligned}
\mathbb{E} \int_0^t Z_+(s) b_-(s) ds &\le (\xi(u) - \xi(u-r)) \int_0^t \mathbb{E} Z_+(s) ds \\
&= (\xi(u) - \xi(u-r)) \int_0^t \left[z_+ + \mathbb{E} \int_0^s b_+(s_1) ds_1 \right] ds \\
&\le (\xi(u) - \xi(u-r)) \left[z_+ t + \frac{t^2}{2} (\xi(u+r) - \xi(u)) \right].
\end{aligned}$$

Similarly,

$$\mathbb{E} \int_0^t Z_-(s) b_+(s) ds \le (\xi(u+r) - \xi(u)) \left[z_- t + \frac{t^2}{2} (\xi(u) - \xi(u-r)) \right].$$

We also note that

$$\frac{1}{m(u,t)}\mathbb{I}_{\{Z_+(t)>0, Z_-(t)>0\}} \geq \frac{1}{2r}\mathbb{I}_{\{Z_+(t)>0, Z_-(t)>0\}},$$

by the definition of $m(u,t)$. Consequently, we obtain

$$\frac{1}{2r}\mathbb{E}\int_0^t \mathbb{I}_{\{Z_+(s)>0, Z_-(s)>0\}} ds \leq \mathbb{E}\int_0^t \frac{\mathbb{I}_{\{Z_+(s)>0, Z_-(s)>0\}}}{m(u,s)} ds \leq \frac{1}{2}G(r,r,u,t),$$

due to (2.6) and the fact that $Z_+(t)Z_-(t) \geq 0$. Thus,

$$\begin{aligned}\int_0^t \mathbb{P}\{m(u,s) < r\} ds &\leq \int_0^t \mathbb{P}\{Z_+(s) > 0, Z_-(s) > 0\} ds \\ &= \mathbb{E}\int_0^t \mathbb{I}_{\{Z_+(s)>0, Z_-(s)>0\}} ds \leq rG(r,r,u,t).\end{aligned}$$

The lemma is proved. $\square$

Corollary 2.3. *For each $\beta > 0$, $u \in (0,1)$ and $t > 0$ the following estimate is true*

$$\mathbb{E}\int_0^t \frac{1}{m^\beta(u,s)} ds \leq \frac{t}{(u \wedge (1-u))^\beta} + \beta \int_0^{u\wedge(1-u)} \frac{1}{r^\beta} G(r,r,u,t) dr,$$

where G is defined by (2.4).

Proof. By Lemma 3.4 [24] and Lemma 2.2, we have

$$\begin{aligned}\mathbb{E}\int_0^t \frac{1}{m^\beta(u,s)} ds &= \int_0^t \mathbb{E}\frac{1}{m^\beta(u,s)} ds = \beta\int_0^t \left(\int_0^\infty r^{\beta-1}\mathbb{P}\left\{\frac{1}{m(u,s)} > r\right\} dr\right) ds \\ &= \beta\int_0^t \left(\int_0^\infty r^{\beta-1}\mathbb{P}\left\{m(u,s) < \frac{1}{r}\right\} dr\right) ds \\ &\leq \beta\int_0^{\frac{1}{u\wedge(1-u)}} \left(\int_0^t r^{\beta-1} ds\right) dr \\ &+ \beta\int_{\frac{1}{u\wedge(1-u)}}^\infty r^{\beta-1}\left(\int_0^t \mathbb{P}\left\{m(u,s) < \frac{1}{r}\right\} ds\right) dr \\ &\leq \frac{t}{(u\wedge(1-u))^\beta} + \beta\int_{\frac{1}{u\wedge(1-u)}}^\infty r^{\beta-1}\frac{1}{r}G\left(\frac{1}{r},\frac{1}{r},u,t\right) dr \\ &= \frac{t}{(u\wedge(1-u))^\beta} + \beta\int_0^{u\wedge(1-u)} \frac{1}{r^\beta}G(r,r,u,t) dr.\end{aligned}$$

The lemma is proved. $\square$

LEMMA 2.3. *For each $T > 0$, $u \in (0,1)$, $r_1 \in (0,u]$, $r_2 \in (0,1-u]$ and $\lambda > 0$*

$$\mathbb{P}\big\{\|X(u+r_2,\cdot) - X(u,\cdot)\|_{C[0,T]} > \lambda,\\ \|X(u,\cdot) - X(u-r_1,\cdot)\|_{C[0,T]} > \lambda\big\} \leq \frac{1}{2\lambda^2} G(r_1, r_2, u, T).$$

PROOF. Let Z_+ and b_+ be defined similarly as in the proof of Lemma 2.2 with r replaced by r_2, and Z_- and b_- with r replaced by r_1. Let

$$\sigma^{\pm} := \inf\{t : \ Z_{\pm}(t) \geq \lambda\} \wedge T$$

and

$$Z_{\pm}^{\sigma^{\pm}}(t) := Z_{\pm}(\sigma^{\pm} \wedge t), \quad t \in [0,T].$$

Then, by Theorem 17.5 [24], Proposition 17.15 ibid. and (2.6), for each $t \geq 0$

$$\mathbb{E} Z_+^{\sigma^+}(t) Z_-^{\sigma^-}(t) + \mathbb{E}\int_0^{t\wedge\sigma^+\wedge\sigma^-} \frac{\mathbb{I}_{\{Z_+(s)>0, Z_-(s)>0\}}}{m(u,s)} ds \\ = z_+ z_- + \mathbb{E}\int_0^{t\wedge\sigma^-} Z_+^{\sigma^+}(s) b_-(s) ds + \mathbb{E}\int_0^{t\wedge\sigma^+} Z_-^{\sigma^-}(s) b_+(s) ds.$$

Similarly as in the proof of Lemma 2.2, we get

$$\mathbb{E} Z_+^{\sigma^+}(T) Z_-^{\sigma^-}(T) \leq \frac{1}{2} G(r_1, r_2, u, T).$$

Next, we note that $Z_+^{\sigma^+}(T) Z_-^{\sigma^-}(T) \geq \lambda^2 \mathbb{I}_{\{\sigma^+ \vee \sigma^- < T\}}$. So,

$$\mathbb{P}\big\{\|X(u+r_2,\cdot) - X(u,\cdot)\|_{C[0,T]} > \lambda, \|X(u,\cdot) - X(u-r_1,\cdot)\|_{C[0,T]} > \lambda\big\} \\ \leq \mathbb{P}\{\sigma^+ \vee \sigma^- < T\} \leq \frac{1}{\lambda^2} \mathbb{E} Z_+^{\sigma^+}(T) Z_-^{\sigma^-}(T) \leq \frac{1}{2\lambda^2} G(r_1, r_2, u, T).$$

The lemma is proved. □

LEMMA 2.4. *For each $\alpha \in (0,1)$ and $t \geq 0$ there exists a constant $C = C(\alpha, t)$ such that for all $r \in (0,1)$ and $u \in [0, r)$ satisfying $r + u \leq 1$ we have*

$$\int_0^t \mathbb{P}\{m(u,s) < r\} ds \leq C e^{C(\xi(1)-\xi(0))^2} \left(\sqrt{u+r}\right)^{\alpha} G_0^{\alpha}(r,u,t),$$

where

$$G_0(r,u,t) = (\xi(u+r) - \xi(u))t + g(u+r) - g(u). \tag{2.7}$$

PROOF. Let $r \in (0,1)$ and $u \in [0,r)$ be fixed. We set

$$Z(t) := X(u+r,t) - X(u,t), \quad t \geq 0,$$

and note that $m(u,t) < r$ implies $Z(t) > 0$. So, in order to prove the lemma, we need to estimate the expectation $\mathbb{E}\int_0^t \mathbb{I}_{\{Z(s)>0\}}ds$.

Let us rewrite Z as follows

$$Z(t) = z_0 + N(t) + \int_0^t b(s)ds, \quad t \geq 0, \tag{2.8}$$

where

$$\begin{aligned} z_0 &: = g(u+r) - g(r), \\ b(t) &: = \xi(u+r) - \xi(u) - \left[\left(\mathrm{pr}_{X_t}\,\xi\right)(u+r) - \left(\mathrm{pr}_{X_t}\,\xi\right)(u)\right] \end{aligned}$$

and N is a continuous square integrable $(\mathcal{F}_t)$-martingale with the quadratic variation

$$\begin{aligned} [N]_t &= [M(u+r,\cdot)]_t + [M(u,\cdot)]_t - 2[M(u+r,\cdot), M(u,\cdot)]_t \\ &= \int_0^t \left(\frac{1}{m(u+r,s)} + \frac{1}{m(u,s)} - \frac{2\mathbb{I}_{\{Z(s)=0\}}}{m(u,s)}\right) ds. \end{aligned}$$

We note that $Z(t) > 0$ implies $m(u,t) = \mathrm{Leb}\{v : \ X(u,t) = X(v,t)\} < u+r$. Thus,

$$[N]_t = \int_0^t a(s)^2 \mathbb{I}_{\{Z(s)>0\}} ds, \quad t \geq 0, \tag{2.9}$$

where $a(t) := \left(\frac{1}{m(u+r,t)} + \frac{1}{m(u,t)}\right)^{\frac{1}{2}} \vee \frac{1}{\sqrt{r+u}} \geq \frac{1}{\sqrt{r+u}}$ for any $t \geq 0$.

Next, we are going to use the Girsanov theorem in order to simplify the term $\int_0^t b(s)ds$ in (2.8). Since the processes Z, a, N, b are functionals of x, where the process x is defined in Corollary 2.1, without loss of generality, we may assume that $\Omega = C([0,\infty), E^n)$, $\mathbb{P} = \mathrm{Law}\{x\}$, $x(t,\omega) = \omega(t)$, $t \geq 0$, $\mathcal{F}$ is the completion of the Borel σ-algebra in $C([0,\infty), E^n)$ and $(\mathcal{F}_t)$ is the right-continuous and complete induced filtration. By Theorem 2.7.1' [21] and (2.9), there exists a Wiener process $w(t)$, $t \geq 0$, on an extended probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ with respect to an extended filtration $\widehat{\mathcal{F}}_t$ such that

$$N(t) = \int_0^t a(s)\mathbb{I}_{\{Z(s)>0\}} dw(s), \quad t \geq 0.$$

Moreover, we can take $\widehat{\Omega} = C([0,\infty), E^n \times \mathbb{R})$ and $(\widehat{\mathcal{F}}_t)$ to be the right-continuous and complete induced filtration on $\widehat{\Omega}$. Let

$$U_t := -\int_0^t \frac{b(s)}{a(s)} dw(s), \quad t \geq 0,$$

and

$$B(t) := w(t) - [w, U]_t = w(t) + \int_0^t \frac{b(s)}{a(s)} ds, \quad t \geq 0.$$

Then, by Novikov's theorem and Lemma 18.18 [24], there exists a probability measure Q on $\widehat{\Omega}$ such that

$$dQ = \exp\left\{ U_t - \frac{1}{2}\int_0^t \frac{b(s)^2}{a(s)^2} ds \right\} d\widehat{\mathbb{P}} \quad \text{on} \quad \widehat{\mathcal{F}}_t$$

for all $t \geq 0$. Using the Girsanov theorem, we have that $B(t)$, $t \geq 0$, is a Wiener process on the probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, Q)$ and, moreover,

$$\begin{aligned}
Z(t) &= z_0 + \int_0^t a(s)\mathbb{I}_{\{Z(s)>0\}} dw(s) + \int_0^t b(s) ds \\
&= z_0 + \int_0^t a(s)\mathbb{I}_{\{Z(s)>0\}} dB(s) + \int_0^t b(s)\mathbb{I}_{\{Z(s)=0\}} ds \\
&= z_0 + \int_0^t a(s)\mathbb{I}_{\{Z(s)>0\}} dB(s) + (\xi(u+r) - \xi(u)) \int_0^t \mathbb{I}_{\{Z(s)=0\}} ds.
\end{aligned}$$

Next, we will consider the process Z on the probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, Q)$ and estimate $\mathbb{E}^Q \int_0^t \mathbb{I}_{\{Z(s)>0\}} ds$, where the expectation $\mathbb{E}^Q$ is taken with respect to the measure Q. We set

$$Y(t) := \sqrt{u+r} Z(t), \quad t \geq 0.$$

It is easily seen that

$$Y(t) = y_0 + \int_0^t \rho(s)\mathbb{I}_{\{Y(s)>0\}} dB(s) + \xi_0 \int_0^t \mathbb{I}_{\{Y(s)=0\}} ds,$$

where $y_0 := \sqrt{u+r} z_0 = \sqrt{u+r}(g(u+r) - g(u))$, $\rho(t) := \sqrt{u+r} a(t) \geq 1$, $t \geq 0$, and $\xi_0 := \sqrt{u+r}(\xi(u+r) - \xi(u))$.

Let

$$R_t := \int_0^t \mathbb{I}_{\{Y(s)>0\}} ds = \int_0^t \mathbb{I}_{\{Z(s)>0\}} ds, \quad t \geq 0.$$

Since

$$1 \le \rho(t) = \sqrt{u+r}a(t) = \left(\frac{u+r}{m(u+r,t)} + \frac{u+r}{m(u,t)}\right)^{\frac{1}{2}} \vee 1$$
$$\le \left(\frac{u+r}{\delta_u \wedge \delta_{u+r}}\right)^{\frac{1}{2}} \vee 1, \quad t \ge 0,$$

where δ_u is defined in Corollary 2.2, we can use Proposition A.1, which is given in the appendix. Thus, we obtain

$$\mathbb{E}^Q R_t \le \sqrt{\frac{2t}{\pi}}(\xi_0 t + y_0).$$

Now, we can estimate $\mathbb{E}R_t = \mathbb{E}\int_0^t \mathbb{I}_{\{Z(s)>0\}}ds$. Since R_t is $\widehat{\mathcal{F}}_t$-measurable, for each $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\begin{aligned}\mathbb{E}R_t &= \mathbb{E}^Q \exp\left\{-U_t + \frac{1}{2}\int_0^t \frac{b(s)^2}{a(s)^2}ds\right\} R_t \\ &\le \left(\mathbb{E}^Q \exp\left\{-pU_t + \frac{p}{2}\int_0^t \frac{b(s)^2}{a(s)^2}ds\right\}\right)^{\frac{1}{p}} \left(\mathbb{E}^Q R_t^q\right)^{\frac{1}{q}} \\ &\le \left(\mathbb{E}\exp\left\{(1-p)U_t - \frac{1-p}{2}\int_0^t \frac{b(s)^2}{a(s)^2}ds\right\}\right)^{\frac{1}{p}} t^{\frac{q-1}{q}} \left(\mathbb{E}^Q R_t\right)^{\frac{1}{q}}\end{aligned}$$

for all $t \ge 0$. In the last inequality, we have applied Jensen's inequality to $R_t^q = \left(\int_0^t \mathbb{I}_{\{Z(s)>0\}}ds\right)^q$. Since $\frac{b(t)^2}{a(t)^2} \le (\xi(1)-\xi(0))^2(u+r) \le (\xi(1)-\xi(0))^2$, $t \ge 0$, and

$$\exp\left\{(1-p)U_t - \frac{(1-p)^2}{2}\int_0^t \frac{b(s)^2}{a(s)^2}ds\right\}, \quad t \ge 0,$$

is a positive martingale with expectation 1, we have

$$\begin{aligned}&\mathbb{E}\exp\left\{(1-p)U_t - \frac{1-p}{2}\int_0^t \frac{b(s)^2}{a(s)^2}ds\right\} \\ &\quad\le e^{\frac{tp(p-1)}{2}(\xi(1)-\xi(0))^2}\mathbb{E}\exp\left\{(1-p)U_t - \frac{(1-p)^2}{2}\int_0^t \frac{b(s)^2}{a(s)^2}ds\right\} \\ &\quad= e^{\frac{tp(p-1)}{2}(\xi(1)-\xi(0))^2}.\end{aligned}$$

Thus,

$$\mathbb{E}R_t \le t^{\frac{q-1}{q}} e^{\frac{t(p-1)}{2}(\xi(1)-\xi(0))^2}\left(\sqrt{\frac{2t}{\pi}}(\xi_0 t + y_0)\right)^{\frac{1}{q}}.$$

Taking $q = \frac{1}{\alpha}$, we obtain

$$\int_0^t \mathbb{P}\{m(u,s) < r\} \leq \mathbb{E}R_t \leq Ce^{C(\xi(1)-\xi(0))^2} \left(\sqrt{u+r}\right)^\alpha G_0^\alpha(r,u,t),$$

with some constant C depending on t and α. The lemma is proved. □

Applying Lemma 2.4 to the process $\{-X(1-u,t), u \in [0,1], t \geq 0\}$, we obtain a similar result for u near 1.

LEMMA 2.5. *For each $\alpha \in (0,1)$ and $t \geq 0$ there exists a constant $C = C(\alpha,t)$ such that for all $r \in (0,1)$ and $u \in (1-r,1]$ satisfying $u - r \geq 0$ we have*

$$\int_0^t \mathbb{P}\{m(u,s) < r\}ds \leq Ce^{C(\xi(1)-\xi(0))^2}(\sqrt{1-u+r})^\alpha G_1^\alpha(r,u,t),$$

where

$$G_1(r,u,t) := (\xi(u) - \xi(u-r))t + g(u) - g(u-r). \tag{2.10}$$

COROLLARY 2.4. *For every $\alpha \in (0,1)$, $\beta > 0$, $u \in \{0,1\}$ and $t > 0$ the estimate*

$$\mathbb{E}\int_0^t \frac{1}{m^\beta(u,s)}ds \leq t + \beta Ce^{C(\xi(1)-\xi(0))^2} \int_0^1 \frac{1}{r^{\beta+1-\frac{\alpha}{2}}} G_u(r,u,t)dr,$$

holds, where G_0 and G_1 are defined in lemmas 2.4 and 2.5 and $C = C(\alpha,t)$ is the same constant as in Lemma 2.4.

PROOF. Using lemmas 2.4 and 2.5, the corollary can be proved similarly as Corollary 2.3. □

PROPOSITION 2.4. *For each $p > 2$ and $0 < \beta < \frac{3}{2} - \frac{1}{p}$ there exists a constant $C = C(p,\beta,t)$ such that*

$$\mathbb{E}\int_0^1 \int_0^t \frac{duds}{m^\beta(u,s)} \leq Ce^{C(\xi(1)-\xi(0))^2}(1 + \|g\|_{L_p}^3 + \|\xi\|_{L_p}).$$

Proof. By Lemma 3.4 [24], we have

$$
\begin{aligned}
&\mathbb{E}\int_0^1\int_0^t \frac{duds}{m^\beta(u,s)} = \beta\int_0^1\int_0^t\int_0^\infty r^{\beta-1}\mathbb{P}\left\{\frac{1}{m(u,s)}>r\right\}dudsdr\\
&= \beta\int_0^2 r^{\beta-1}\left(\int_0^1\int_0^t \mathbb{P}\left\{m(u,s)<\frac{1}{r}\right\}duds\right)dr\\
&+ \beta\int_2^\infty r^{\beta-1}\left(\int_0^{\frac{1}{r}}\int_0^t \mathbb{P}\left\{m(u,s)<\frac{1}{r}\right\}duds\right)dr\\
&+ \beta\int_2^\infty r^{\beta-1}\left(\int_{\frac{1}{r}}^{1-\frac{1}{r}}\int_0^t \mathbb{P}\left\{m(u,s)<\frac{1}{r}\right\}duds\right)dr\\
&+ \beta\int_2^\infty r^{\beta-1}\left(\int_{1-\frac{1}{r}}^{1}\int_0^t \mathbb{P}\left\{m(u,s)<\frac{1}{r}\right\}duds\right)dr =: I_1+I_2+I_3+I_4.
\end{aligned}
$$

The first integral $I_1 \le 2^\beta t$, since trivially $\mathbb{P}\left\{m(u,s)<\frac{1}{r}\right\}\le 1$.

Next, using Lemma 2.4, we estimate I_2. Let $\alpha\in(0\vee(2\beta-2),1)$ be fixed and satisfy $1+\frac{\alpha}{2}-\frac{\alpha}{p}>\beta$. Then for each $r' := \frac{1}{r}\le\frac{1}{2}$

$$
\begin{aligned}
&\int_0^{r'}\int_0^t \mathbb{P}\left\{m(u,s)<r'\right\}duds\\
&\qquad\le C_1\int_0^{r'}\left(\sqrt{u+r'}\right)^\alpha\left[(\xi(u+r')-\xi(u))t+g(u+r')-g(u)\right]^\alpha du\\
&\qquad\le C_1t^\alpha\int_0^{r'}\left(\sqrt{u+r'}\right)^\alpha(\xi(u+r')-\xi(u))^\alpha du\\
&\qquad+ C_1\int_0^{r'}\left(\sqrt{u+r'}\right)^\alpha(g(u+r')-g(u))^\alpha du,
\end{aligned}
$$

where $C_1 = C_1(\alpha,t) := C(\alpha,t)e^{C(\alpha,t)(\xi(1)-\xi(0))^2}$. We can easily estimate the first term as follows

$$
C_1t^\alpha\int_0^{r'}\left(\sqrt{u+r'}\right)^\alpha(\xi(u+r')-\xi(u))^\alpha du \le 2^{\frac{\alpha}{2}}C_1t^\alpha(r')^{1+\frac{\alpha}{2}}(\xi(1)-\xi(0))^\alpha,
$$

using $\xi(u+r')-\xi(u)\le\xi(1)-\xi(0)$ and $u+r'\le 2r'$. The second term will be estimated using Hölder's inequality. So, for q satisfying $\frac{1}{p}+\frac{1}{q}=1$ we have

$$
\begin{aligned}
C_1\int_0^{r'}\left(\sqrt{u+r'}\right)^{\alpha}(g(u+r')-g(u))^{\alpha}du &\leq C_1(2r')^{\frac{\alpha}{2}}\int_0^{r'}(g(u+r')-g(u))^{\alpha}du\\
&\leq C_1(2r')^{\frac{\alpha}{2}}(r')^{1-\alpha}\left[\int_0^{r'}(g(u+r')-g(u))du\right]^{\alpha}\\
&\leq 2^{\frac{\alpha}{2}}C_1(r')^{1-\frac{\alpha}{2}}\left[\int_0^1\left(\mathbb{I}_{[r',2r']}(u)-\mathbb{I}_{[0,r']}(u)\right)g(u)du\right]^{\alpha}\\
&\leq 2^{\frac{\alpha}{2}}C_1(r')^{1-\frac{\alpha}{2}}\|g\|_{L_p}^{\alpha}\left[\int_0^1\left|\mathbb{I}_{[r',2r']}(u)-\mathbb{I}_{[0,r']}(u)\right|^q du\right]^{\frac{\alpha}{q}}\\
&= 2^{\frac{\alpha}{2}}C_1(r')^{1-\frac{\alpha}{2}}\|g\|_{L_p}^{\alpha}(2r')^{\frac{\alpha}{q}} = 2^{\frac{\alpha}{2}+\frac{\alpha}{q}}C_1(r')^{1+\frac{\alpha}{q}-\frac{\alpha}{2}}\|g\|_{L_p}^{\alpha}.
\end{aligned}
$$

Thus,

$$
I_2 \leq 2^{\frac{\alpha}{2}}C_1t^{\alpha}(\xi(1)-\xi(0))^{\alpha}\int_2^{\infty}r^{\beta-1-1-\frac{\alpha}{2}}dr+2^{\frac{\alpha}{2}+\frac{\alpha}{q}}C_1\|g\|_{L_p}^{\alpha}\int_2^{\infty}r^{\beta-1-1-\frac{\alpha}{q}+\frac{\alpha}{2}}dr,
$$

where $\int_2^{\infty}r^{\beta-1-1-\frac{\alpha}{2}}dr$ and $\int_2^{\infty}r^{\beta-1-1-\frac{\alpha}{q}+\frac{\alpha}{2}}dr$ are finite because $\beta-2-\frac{\alpha}{2}<-1$ and $\beta-2-\frac{\alpha}{q}+\frac{\alpha}{2}<-1$, by the choice of α.

Similarly, we obtain the same estimate for I_4, by Lemma 2.5.

In order to estimate I_3, we use Lemma 2.2. So, for $r'=\frac{1}{r}\leq\frac{1}{2}$ we get

$$
\begin{aligned}
\int_{r'}^{1-r'}\int_0^t\mathbb{P}\left\{m(u,s)<r'\right\}duds &\leq \int_{r'}^{1-r'}r'G(r',u,t)du\\
&= 2r'\int_{r'}^{1-r'}(g(u+r')-g(u))(g(u)-g(u-r'))du\\
&+ 2tr'\int_{r'}^{1-r'}(\xi(u)-\xi(u-r'))(g(u+r')-g(u))du\\
&+ 2t^2r'\int_{r'}^{1-r'}(\xi(u)-\xi(u-r'))(\xi(u+r')-\xi(u))du\\
&+ 2tr'\int_{r'}^{1-r'}(\xi(u+r')-\xi(u))(g(u)-g(u-r'))du\\
&=: J_1+J_2+J_3+J_4,
\end{aligned}
$$

where G is defined by (2.4). First, we estimate J_1. Using the trivial inequality $x^2\leq x+x^2\mathbb{I}_{\{x>1\}}$, $x\geq 0$, and Hölder's inequality with $\frac{1}{l}+\frac{1}{l'}=1$ and $l=\frac{p}{2}$, we obtain

$$
\begin{aligned}
J_1 &\le 2r' \int_{r'}^{1-r'} (g(u+r') - g(u-r'))^2 du \\
&\le 2r' \int_{r'}^{1-r'} (g(u+r') - g(u-r')) du \\
&+ 2r' \int_{r'}^{1-r'} (g(u+r') - g(u-r'))^2 \mathbb{I}_{\{g(u+r')-g(u-r')>1\}} du \\
&\le 2r' \int_{r'}^{1-r'} (g(u+r') - g(u-r')) du \\
&+ 2r' \left[\int_{r'}^{1-r'} (g(u+r') - g(u-r'))^p du \right]^{\frac{2}{p}} \left[\int_{r'}^{1-r'} \mathbb{I}_{\{g(u+r')-g(u-r')>1\}} du \right]^{\frac{1}{l'}} \\
&\le 2r' \int_{r'}^{1-r'} (g(u+r') - g(u-r')) du \\
&+ 8r' \|g\|_{L_p}^2 \left[\int_{r'}^{1-r'} (g(u+r') - g(u-r')) du \right]^{1-\frac{2}{p}}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\int_{r'}^{1-r'} (g(u+r') - g(u-r')) du &= \int_0^1 \left(\mathbb{I}_{[2r',1]} - \mathbb{I}_{[0,1-2r']} \right) g(u) du \\
&\le \|g\|_{L_p} \left[\int_0^1 \left| \mathbb{I}_{[2r',1]} - \mathbb{I}_{[0,1-2r']} \right| du \right]^{1-\frac{1}{p}} = \|g\|_{L_p} (4r')^{1-\frac{1}{p}},
\end{aligned} \tag{2.11}
$$

J_1 can be estimated as follows

$$
J_1 \le c_1 \|g\|_{L_p} (r')^{2-\frac{1}{p}} + c_2 \|g\|_{L_p}^{3-\frac{2}{p}} (r')^{2+\frac{2}{p^2}-\frac{3}{p}},
$$

where c_1, c_2 are constants. Using (2.11), we have

$$
\begin{aligned}
J_2 + J_3 + J_4 &\le c_3 t (r')^{2-\frac{1}{p}} (\xi(1) - \xi(0)) \|g\|_{L_p} \\
&+ 2t^2 r' (\xi(1) - \xi(0)) \int_{r'}^{1-r'} (\xi(u+r') - \xi(u-r')) du \\
&\le c_3 t (r')^{2-\frac{1}{p}} (\xi(1) - \xi(0)) \|g\|_{L_p} + c_4 t^2 (r')^{2-\frac{1}{p}} (\xi(1) - \xi(0)) \|\xi\|_{L_p}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
I_3 &\leq \beta C_2(\xi(1)-\xi(0))\|g\|_{L_p}\int_2^\infty r^{\beta-1-2+\frac{1}{p}}dr\\
&+C_3(\xi(1)-\xi(0))\|g\|_{L_p}^{3-\frac{2}{p}}\int_2^\infty r^{\beta-1-2-\frac{2}{p^2}+\frac{3}{p}}dr\\
&+C_4(\xi(1)-\xi(0))\|\xi\|_{L_p}\int_2^\infty r^{\beta-1-2+\frac{1}{p}}dr,
\end{aligned}
$$

where constants C_i, $i\in\{2,3,4\}$, only depend on p and t and the integrals are finite according to the choice of β. The lemma is proved.

□

LEMMA 2.6. *For each $t>0$, $\delta\in[0,1)$ and $\varepsilon>\frac{2\delta}{1-\delta}$ there exists a constant $C=C(t,\delta,\varepsilon)$ such that*

$$
\mathbb{E}\sup_{s\in[0,t]}\|X_s-g\|_{L_{2+\delta}}^{2+\delta}\leq Ce^{C(\xi(1)-\xi(0))^2}(1+\|g\|_{L_{2+\varepsilon}}^3+\|\xi\|_{L_{2+\varepsilon}}).
$$

PROOF. By the Burkholder-Davis-Gundy inequality, $(R3)$, $(R4)$ and Proposition 2.4, we have

$$
\begin{aligned}
\mathbb{E}\sup_{s\in[0,t]}\|X_s-g\|_{L_{2+\delta}}^{2+\delta} &= \mathbb{E}\sup_{s\in[0,t]}\int_0^1|X(u,s)-g(u)|^{2+\delta}du\\
&\leq \int_0^1\mathbb{E}\sup_{s\in[0,t]}|X(u,s)-g(u)|^{2+\delta}du\leq 2^{1+\delta}\int_0^1\mathbb{E}\sup_{s\in[0,t]}|M(u,s)|^{2+\delta}du\\
&+2^{1+\delta}\int_0^1\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s(\xi(u)-\left(\mathrm{pr}_{X_r}\xi\right)(u))dr\right|^{2+\delta}du\\
&\leq 2^{1+\delta}C_1\int_0^1\mathbb{E}\left(\int_0^t\frac{ds}{m(u,s)}\right)^{1+\frac{\delta}{2}}du+2^{1+\delta}t^{2+\delta}(\xi(1)-\xi(0))^{2+\delta}\\
&\leq 2^{1+\delta}C_1t^{\frac{\delta}{2}}\int_0^1\left(\mathbb{E}\int_0^t\frac{ds}{m(u,s)^{1+\frac{\delta}{2}}}\right)du+2^{1+\delta}t^{2+\delta}(\xi(1)-\xi(0))^{2+\delta}\\
&\leq 2^{1+\delta}C_1t^{\frac{\delta}{2}}Ce^{C(\xi(1)-\xi(0))^2}(1+\|g\|_{L_{2+\varepsilon}}^3+\|\xi\|_{L_{2+\varepsilon}})\\
&+2^{1+\delta}t^{2+\delta}(\xi(1)-\xi(0))^{2+\delta},
\end{aligned}
$$

where C_1 depends on δ. We could apply Proposition 2.4 in the latter inequality, since $1+\frac{\delta}{2}<\frac{3}{2}-\frac{1}{2+\varepsilon}$. The lemma is proved. □

COROLLARY 2.5. *Under the assumptions of Lemma 2.6,*

$$\mathbb{E} \sup_{s\in[0,t]} \|X_s\|_{L_{2+\delta}}^{2+\delta} \leq Ce^{C(\xi(1)-\xi(0))^2}(1+\|g\|_{L_{2+\delta}}^{2+\delta}+\|g\|_{L_{2+\varepsilon}}^3+\|\xi\|_{L_{2+\varepsilon}}),$$

where C depends on t, δ and ε.

3. Tightness results. Let $\{\xi_n,\ n \geq 1\}$ and $\{g_n,\ n \geq 1\}$ be arbitrary sequences in $\mathcal{S}^{\uparrow}$ and let $\{X^n,\ n \geq 1\}$ be a sequence of random elements in $D([0,1], C([0,\infty)))$ satisfying $(R1)-(R4)$ with ξ_n, g_n instead of ξ, g. Here, we also assume that such elements exist. Let $M^n(u,\cdot)$ and $A^n(u,\cdot)$ denote the martingale part and the part of bounded variation of $X^n(u,\cdot)$ for every $u \in [0,1]$, that is,

$$A^n(u,t) = \int_0^t (\xi_n(u) - \left(\mathrm{pr}_{X_s^n}\,\xi_n\right)(u))ds$$

and

$$M^n(u,t) = X^n(u,t) - g_n(u) - A^n(u,t),$$

for all $t \geq 0$, where $X_t^n := X^n(\cdot,t)$, $t \geq 0$.

3.1. *Tightness of weak solutions.* In this section, we check the tightness of the family $\{X_\cdot^n,\ n \geq 1\}$, where we consider X^n as random processes in $L_2^{\uparrow}$. Let

$$M_t^n := M^n(\cdot,t) \quad \text{and} \quad A_t^n := A^n(\cdot,t), \quad t \geq 0, \quad n \geq 1.$$

We will also consider $M_\cdot^n$, $n \geq 1$, and $A_\cdot^n$, $n \geq 1$, as stochastic processes taking values in L_2. Using $(R1)-(R4)$, one can show that the processes X_t^n, $t \geq 0$, are weak solutions to SDE (1.3) with g and ξ replaced by g_n and ξ_n.

In the next section, we will show that each limit point of $\{X_\cdot^n,\ n \geq 1\}$ (which will exist by the tightness and Prokhorov's theorem) is a weak solution to SDE (1.3). For this, we will need the convergence of the martingale parts, the parts of bounded variation and the quadratic variation processes. So, let $\{e_i,\ i \in \mathbb{N}\}$ be a fixed orthonormal basis of L_2 and $M^n(e_i) := (M_\cdot^n, e_i)$, $i \in \mathbb{N}$. We are going to prove that the family

$$\overline{X}^n := \left(M^n, A^n, ([M^n(e_i), M^n(e_j)])_{(i,j)\in\mathbb{N}^2}, \langle M^n\rangle\right), \quad n \geq 1, \tag{3.1}$$

is tight in

$$\mathcal{W} := C([0,\infty), L_2) \times C([0,\infty), L_2) \times C([0,\infty))^{\mathbb{N}^2} \times C([0,\infty))$$

under the assumptions that $\{g_n,\ \xi_n\ n \geq 1\}$ is bounded in $L_{2+\delta}$ for some $\delta > 0$ and $\{\xi_n(1) - \xi_n(0),\ n \geq 1\}$ is bounded in $\mathbb{R}$. In order to prove the tightness, we need the following lemma.

LEMMA 3.1. *For every $C > 0$ and $\delta > 0$ the set $K_C := \{g :\in L_2^{\uparrow} : \|g\|_{L_{2+\delta}} \leq C\}$ is compact in $L_2^{\uparrow}$ and, consequently, in L_2.*

For the proof of the lemma see e.g. Lemma 5.1 [28].

PROPOSITION 3.1. *If there exists $\delta > 0$ such that $\{g_n,\ \xi_n,\ n \geq 1\}$ is bounded in $L_{2+\delta}$ and $\{\xi_n(1) - \xi_n(0),\ n \geq 1\}$ is bounded in $\mathbb{R}$, then $\{\overline{X}^n,\ n \geq 1\}$ is tight in $\mathcal{W}$.*

PROOF. In order to prove the proposition, it is enough to show that the coordinate processes of $\{\overline{X}^n,\ n \geq 1\}$ are tight in the corresponding spaces, by Proposition 3.2.4 [16]. So, we first prove that $\{A^n,\ n \geq 1\}$ is tight in $C([0,\infty), L_2)$. We note that $A_t^n = t\xi_n - \int_0^t \mathrm{pr}_{X_s^n} \xi_n ds$ and the sequence $\{t\xi_n,\ t \geq 0\}_{n \geq 1}$ is relatively compact in $C([0,\infty), L_2)$, by Lemma 3.1. Thus, to prove the tightness of $\{A^n,\ n \geq 1\}$, it is enough to show that $\left\{\widehat{A}_\cdot^n := \int_0^\cdot \mathrm{pr}_{X_s^n} \xi_n ds,\ n \geq 1\right\}$ is tight in $C([0,\infty), L_2)$. Since $\mathrm{pr}_{X_t^n} \xi_n$ belongs to $L_2^{\uparrow}$ for each $t \geq 0$ (see Lemma A.2), the process $\widehat{A}^n$ takes values in $L_2^{\uparrow}$ for all $n \geq 1$. Hence, the tightness follow from Jakubowski's tightness criterion [23], Lemma 3.1 and the estimate

(3.2)

$$
\begin{aligned}
\mathbb{E} \sup_{s \in [0,t]} \left\| \int_0^s \mathrm{pr}_{X_r^n} \xi_n dr \right\|_{L_{2+\delta}}^{2+\delta} &= \mathbb{E} \sup_{s \in [0,t]} \int_0^1 \left| \int_0^s (\mathrm{pr}_{X_r^n} \xi_n)(u) dr \right|^{2+\delta} du \\
&\leq t^{1+\delta} \mathbb{E} \int_0^t \left(\int_0^1 |\mathrm{pr}_{X_r^n} \xi_n|^{2+\delta}(u) du \right) dr \\
&\leq t^{1+\delta} \mathbb{E} \int_0^t \left(\int_0^1 \left(\mathrm{pr}_{X_r^n} |\xi_n|^{\frac{2+\delta}{2}} \right)^2 (u) du \right) dr \\
&= t^{1+\delta} \mathbb{E} \int_0^t \left\| \mathrm{pr}_{X_r^n} |\xi_n|^{\frac{2+\delta}{2}} \right\|_{L_2}^2 dr \leq t^{2+\delta} \|\xi_n\|_{L_{2+\delta}}^{2+\delta},
\end{aligned}
$$

where the inequality $|\mathrm{pr}_{X_r^n} \xi_n|^{\frac{2+\delta}{2}} \leq \mathrm{pr}_{X_r^n} |\xi_n|^{\frac{2+\delta}{2}}$ follows from Remark A.1 (ii) and Hölder's inequality for conditional expectations. Indeed, by (3.2) and Lemma 3.1, for each $\varepsilon > 0$ and $T > 0$ there exists $C > 0$ such that

$$
\mathbb{P}\left\{ \exists t \in [0,T],\ \widehat{A}_t^n \notin K_C \right\} < \varepsilon
$$

for all $n \geq 1$, where $K_C = \{g :\in L_2^{\uparrow} :\ \|g\|_{L_{2+\delta}} \leq C\}$ is compact in L_2. Moreover, for every $h \in L_2$ the sequence $\left\{ (\widehat{A}_\cdot^n, h)_{L_2},\ n \geq 1 \right\}$ is tight in

$C([0,\infty))$, by the Aldous tightness criterion (see e.g. Theorem 3.6.5. [7]). This implies the tightness of $\{\widehat{A}^n,\ n \geq 1\}$.

Similarly, we can prove that $\{X^n,\ n \geq 1\}$ is tight in $C([0,\infty), L_2^{\uparrow})$, using additionally Corollary 2.5. Hence, by Proposition 3.2.4 [16] $\{(X^n, A^n),\ n \geq 1\}$ is tight in $C([0,\infty), L_2)^2$ and, consequently, the sequence of processes $\{M^n,\ n \geq 1\}$, which are defined as $M^n_t = X^n_t - g_n - A^n_t$, $t \geq 0$, is tight in $C([0,\infty), L_2)$.

The tightness of $[M^n(e_i), M^n(e_j)]$ in $C([0,\infty))$ easily follows from the Aldous tightness criterion and the estimate

$$|(\operatorname{pr}_{X^n_t} e_i, e_j)_{L_2}| \leq \| \operatorname{pr}_{X^n_t} e_i\|_{L_2} \|e_j\|_{L_2} \leq 1$$

for all $t \geq 0$ and $i, j, n \in \mathbb{N}$.

Next, we prove the tightness of $\{\langle M^n \rangle,\ n \geq 1\}$ in $C([0,\infty))$. We are going to use the Aldous tightness criterion again. So, by Lemma A.3 and Proposition 2.4,

$$\begin{aligned}\mathbb{E}\langle M^n\rangle_t = \mathbb{E}\int_0^t \| \operatorname{pr}_{X^n_s} \|^2_{HS} ds = \mathbb{E}\int_0^t \left(\int_0^1 \frac{du}{m_n(u,s)} \right) ds \\ \leq C e^{C(\xi_n(1)-\xi_n(0))^2}(1 + \|g_n\|^3_{L_{2+\delta}} + \|\xi_n\|_{L_{2+\delta}}),\end{aligned}$$

where the constant C depends on t and δ. Thus, the boundedness of $\{g_n,\ \xi_n,\ n \geq 1\}$ in $L_{2+\delta}$ and $\{\xi_n(1) - \xi_n(0),\ n \geq 1\}$ in $\mathbb{R}$ yields the tightness of $\langle M^n \rangle_t$ in $\mathbb{R}$ for all $t \geq 0$. Next, let $T > 0$, $\{r_n,\ n \geq 1\} \subset [0,T]$ be any sequence decreasing to 0 and $\{\tau_n,\ n \geq 1\}$ be any sequence of $(\mathcal{F}^{X^n}_t)$-stopping times on $[0,T]$. Then for each $\varepsilon > 0$ and $\beta \in \left(1, \frac{3}{2} - \frac{1}{2+\delta}\right)$

$$\begin{aligned}\mathbb{P}\big\{|\langle M^n\rangle_{\tau_n+r_n} - \langle M^n\rangle_{\tau_n}| > \varepsilon\big\} &\leq \frac{1}{\varepsilon}\mathbb{E}\left(\langle M^n\rangle_{\tau_n+r_n} - \langle M^n\rangle_{\tau_n}\right) \\ &= \frac{1}{\varepsilon}\mathbb{E}\int_{\tau_n}^{\tau_n+r_n} \| \operatorname{pr}_{X^n_s} \|^2_{HS} ds \\ &\overset{\text{L. A.3}}{=} \frac{1}{\varepsilon}\mathbb{E}\int_{\tau_n}^{\tau_n+r_n} \left(\int_0^1 \frac{du}{m_n(u,s)} \right) ds \\ &\leq \frac{r_n^{\frac{\beta-1}{\beta}}}{\varepsilon}\mathbb{E}\int_0^1 \int_0^{2T} \frac{duds}{m_n^\beta(u,s)}.\end{aligned}$$

Consequently, $\mathbb{P}\{|\langle M^n\rangle_{\tau_n+r_n} - \langle M^n\rangle_{\tau_n}| > \varepsilon\} \to 0$ as $n \to \infty$, by Proposition 2.4. Thus, the Aldous tightness criterion implies the compactness of $\{\langle M^n \rangle,\ n \geq 1\}$ in $C([0,\infty))$. This finishes the proof of the proposition. □

Corollary 3.1. *Let $\delta > 0$, $\{g_n,\ \xi_n,\ n \geq 1\} \subset \mathcal{S}^{\uparrow}$ and $\{\overline{X}^n,\ n \geq 1\}$ be defined by (3.1). If $g_n \to g$, $\xi_n \to \xi$ in $L_{2+\delta}$ and $\{\xi_n(1) - \xi_n(0),\ n \geq 1\}$ is bounded, then there exists a subsequence $N \subseteq \mathbb{N}$ and a random element $\overline{X}$ in $\mathcal{W}$ such that $\overline{X}^n \to \overline{X}$ in $\mathcal{W}$ in distribution along N.*

Proof. The statement of the corollary follows from Prochorov's theorem and Proposition 3.1. □

3.2. *Tightness in the Skorohod space.* In this subsection, we will consider the processes M^n, A^n, X^n, which were defined at the beginning of Section 3, as random elements in the Skorohod space. If $\pi = [a, b] \subset [0, 1]$, then we will set

$$X^{\pi,n}(u, \cdot) = \begin{cases} X^n(u, \cdot), & u \in [a, b), \\ \lim_{u \uparrow b} X^n(u, \cdot), & u = b, \end{cases}$$

where $\lim_{u\uparrow b} X^n(u, \cdot)$ exists in $C([0, \infty))$, e.g. by Corollary 2.1. Let G^n be defined by (2.4) with g and ξ replaced by g_n and ξ_n, respectively, and let

$$\begin{aligned} G^n_a(r, t) &:= (\xi_n(a + r) - \xi_n(a))t + g_n(a + r) - g_n(a), \quad r \in (0, 1 - a], \\ G^n_b(r, t) &:= (\xi_n(b-) - \xi_n(b - r))t + g_n(b-) - g_n(b - r), \quad r \in (0, b]. \end{aligned}$$

Proposition 3.2. *Let $T > 0$, $\pi := [a, b] \subset [0, 1]$ and $\{g_n,\ \xi_n,\ n \geq 1\}$ be bounded in $L^{\uparrow}_{\infty}$. If there exist $\beta > 0$ and $C > 0$ such that for each $n \geq 1$*

(c1) $G^n(r \wedge (u - a), r \wedge (b - u), u, T) \leq Cr^{1+\beta}$ for all $u \in (a, b),\ r > 0$;
(c2) $G^n_v(r, T) \leq Cr^{\frac{1}{2}+\beta}$ for all $v \in \{a, b\}$ and $r \in (0, b \wedge 1 - a]$,

then the family $\{X^{\pi,n}(u, t),\ u \in [a, b],\ t \in [0, T]\}_{n\geq1}$ is tight in $D([a, b], C([0, T]))$.

Proof. The proof is similar to the proof of Proposition 2.2 [33]. Here, we indicate the main steps only.

The statement will follow from theorems 3.8.6 and 3.8.8 [16] and Remark 3.8.9 ibid. We only have to check the following properties of $\{X^{\pi,n},\ n \geq 1\}$.

(a) There exists $C_1 > 0$ such that

$$\begin{aligned} \mathbb{P}\big\{ &\|X^{\pi,n}((u + r) \wedge b, \cdot) - X^{\pi,n}(u, \cdot)\|_{C[0,T]} > \lambda, \\ &\|X^{\pi,n}(u, \cdot) - X^{\pi,n}((u - r) \vee a, \cdot)\|_{C[0,T]} > \lambda\big\} \leq \frac{C_1 r^{1+\beta}}{\lambda^2} \end{aligned}$$

for all $n \in \mathbb{N}$, $u \in (a, b)$, $r > 0$ and $\lambda > 0$.

(b) For each $\alpha > 1$

$$\lim_{\delta\to 0+}\sup_{n\geq 1}\mathbb{E}\left[\|X^{\pi,n}(a+\delta,\cdot)-X^{\pi,n}(a,\cdot)\|^{\alpha}_{C([0,T])}\wedge 1\right]=0 \tag{3.3}$$

and

$$\lim_{\delta\to 0+}\sup_{n\geq 1}\mathbb{E}\left[\|X^{\pi,n}(b,\cdot)-X^{\pi,n}(b-\delta,\cdot)\|^{\alpha}_{C([0,T])}\wedge 1\right]=0. \tag{3.4}$$

(c) For all $u\in[a,b]$ the sequence $\{X^{\pi,n}(u,t),\ t\in[0,T]\}_{n\geq 1}$ is tight in $C([0,T])$.

Properties (a), (b) and (c) are needed for the verification of conditions (8.39), (8.30)[10] of [16] and (a) of Theorem 3.7.2 ibid., respectively.

Property (a) immediately follows from $(c1)$ and Lemma 2.3.

Next, let us prove (b). We check only (3.4). Using the monotonisity of $X^{\pi,n}(u,\cdot)$, $u\in[a,b]$, and the monotone convergence theorem, we have for each $\delta\in(0,b-a)$

$$\begin{aligned}\mathbb{E}\Big[&\|X^{\pi,n}(b,\cdot)-X^{\pi,n}(b-\delta,\cdot)\|^{\alpha}_{C([0,T])}\wedge 1\Big]\\ &=\sup_{\gamma\in(0,\delta)}\mathbb{E}\Big[\|X^{\pi,n}(b-\gamma,\cdot)-X^{\pi,n}(b-\delta,\cdot)\|^{\alpha}_{C([0,T])}\wedge 1\Big]\\ &=\sup_{\gamma\in(0,\delta)}\mathbb{E}\Big[\|X^{n}(b-\gamma,\cdot)-X^{n}(b-\delta,\cdot)\|^{\alpha}_{C([0,T])}\wedge 1\Big].\end{aligned} \tag{3.5}$$

We set for fixed $\delta\in(0,b-a)$ and $\gamma\in(0,\delta)$

$$X^n_{\delta,\gamma}(t):=g_n(b-\gamma)-g_n(b-\delta)+M^n_{\delta,\gamma}(t)+A^n_{\delta,\gamma}(t),\quad t\in[0,T],$$

where

$$\begin{aligned}M^n_{\delta,\gamma}(t):&=M^n(b-\gamma,t)-M^n(b-\delta,t),\\ A^n_{\delta,\gamma}(t):&=\int_0^t(b^n_{\delta,\gamma}(s)\vee 0)ds\end{aligned}$$

and

$$b^n_{\delta,\gamma}(s):=\xi_n(b-\gamma)-\xi_n(b-\delta)-\left[\left(\mathrm{pr}_{X^n_s}\,\xi_n\right)(b-\gamma)-\left(\mathrm{pr}_{X^n_s}\,\xi_n\right)(b-\delta)\right].$$

[10] Here, we used the statement for the tightness in $D([a,\infty),C([0,T]))$, which can be applied to $\{X^{\pi,n}(u\wedge b,t),\ u\in[0,\infty),\ t\in[0,T]\}_{n\geq 1}$. Since $D([a,b],C([0,T]))$ contains functions which are continuous in b, additional property (3.4) is needed for the tightness there.

We also introduce the stopping time

$$\sigma^n_{\delta,\gamma} := \inf\left\{t:\ X^n_{\delta,\gamma}(t) = 1\right\} \wedge T.$$

Since $M^n_{\delta,\gamma}$ is a continuous martingale and $A^n_{\delta,\gamma}$ is an increasing and continuous process, $X^n_{\delta,\gamma}$ is a continuous submartingale. Moreover,

$$0 \leq X^n(b-\gamma,t) - X^n(b-\delta,t) \leq X^n_{\delta,\gamma}(t)$$

for all $t \in [0,T]$. Thus, by Doob's martingale inequality (see e.g. Proposition 2.2.16 [16]) and the estimate

$$b^n_{\delta,\gamma}(t) \vee 0 \leq \xi_n(b-\gamma) - \xi_n(b-\delta), \quad t \in [0,T],$$

we can estimate

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} (X^n(b-\gamma,t) - X^n(b-\delta,t))^\alpha \wedge 1\right] &\leq \mathbb{E}\left[\sup_{t\in[0,T]} (X^n_{\delta,\gamma}(t \wedge \sigma^n_{\delta,\gamma}))^\alpha\right] \\
&\leq C_\alpha \mathbb{E}\left[(X^n_{\delta,\gamma}(T \wedge \sigma^n_{\delta,\gamma}))^\alpha\right] \leq C_\alpha \mathbb{E}\left[X^n_{\delta,\gamma}(T \wedge \sigma^n_{\delta,\gamma})\right] \\
&\leq C_\alpha[g_n(b-\gamma) - g_n(b-\delta) + T(\xi_n(b-\gamma) - \xi_n(b-\delta))],
\end{aligned}$$

where $C_\alpha = \left(\frac{\alpha}{\alpha-1}\right)^\alpha$. Thus, by $(c2)$ and (3.5),

$$\mathbb{E}\left[\|X^{\pi,n}(b,\cdot) - X^{\pi,n}(b-\delta,\cdot)\|^\alpha_{C([0,T])} \wedge 1\right] \leq C_\alpha G^n_b(\delta) \leq C_\alpha C \delta^\beta.$$

This implies (3.4).

Property (c) can be proved similarly as Lemma 2.5 [33], using the Aldous tightness criterion and the estimates

$$\mathbb{E}\int_0^T \frac{dt}{m_n^{1+\frac{\beta}{2}}(u,t)} \leq \tilde{C} < \infty, \tag{3.6}$$

$$\begin{aligned}
\mathbb{E}|X^{\pi,n}(u,t)| &\leq \mathbb{E}\left|X^{\pi,n}(u,t) - \int_0^1 X^n(v,t)dv\right| + \mathbb{E}\left|\int_0^1 X^n(v,t)dv\right| \\
&\leq \mathbb{E}(X^n(1,t) - X^n(0,t)) + \mathbb{E}\left|\int_0^1 X^n(v,t)dv\right| \\
&\leq g_n(1) - g_n(0) + T(\xi_n(1) - \xi_n(0)) + \mathbb{E}\left|\int_0^1 X^n(v,t)dv\right|
\end{aligned}$$

for all $n \geq 1$, where $\int_0^1 X^n(v,t)dv = (X_t^n, 1)_{L_2}$, $t \in [0,T]$, is a Wiener process, since it is a continuous martingale with the quadratic variation $\int_0^t \mathrm{pr}_{X_s^n} 1ds = t$ (we note that $(A_t^n, 1)_{L_2} = 0$). Here, (3.6) follows from $(c2)$ and corollaries 2.3, 2.4. The proposition is proved. $\square$

REMARK 3.1. *If* $(c1)$, $(c2)$ *hold for some* $T > 0$, *then one can easily check that they hold for any* $T \geq 0$, *with* C *depending on* T. *Thus, under the assumptions of Proposition 3.2 (for some* $T > 0$*), the family* $\{X^{\pi,n}(u,t),\ u \in [a,b],\ t \in [0,\infty)\}_{n\geq 1}$ *is tight in* $D([a,b], C([0,\infty)))$.

4. Properties of limit points.

4.1. *Identification of the limit.* In this section, we assume that $\{g_n,\ n \geq 1\}$, $\{\xi_n,\ n \geq 1\}$ be arbitrary sequences of functions from $L_2^{\uparrow}$ (not necessarily from $\mathcal{S}^{\uparrow}$) and processes X^n, $n \geq 1$, defined on the same probability space, are weak solutions to SDE (1.3) with initial conditions g_n and interacting potentials ξ_n. For each $n \geq 1$ we denote $M^n := M^{X^n}$, $A^n := A^{X^n}$ and recall that the process

$$X_t^n = g_n + M_t^n + A_t^n, \quad t \geq 0,$$

takes values in $L_2^{\uparrow}$,

$$A_t^n = \int_0^t (\xi_n - \mathrm{pr}_{X_s^n} \xi_n) ds$$

and M^n is a continuous square integrable $(\mathcal{F}_t^{X^n})$-martingale in L_2 with the quadratic variation process

$$\langle\langle M^n \rangle\rangle_t = \int_0^t \mathrm{pr}_{X_s^n} ds,$$

and the increasing process

$$\langle M^n \rangle_t = \int_0^t \| \mathrm{pr}_{X_s^n} \|_{HS}^2 ds.$$

Let $\{e_i,\ i \in \mathbb{N}\}$ be a fixed orthonormal basis of L_2 and $\overline{X}^n$ be defined by (3.1).

The following theorem is the main result of this section, which states that a limit of solution to equation (1.3) is again a solution to (1.3).

THEOREM 4.1. *Let* $\{g_n,\ n \geq 1\}$, $\{\xi_n,\ n \geq 1\}$ *converge to* g *and* ξ *in* L_2, *respectively, and let the sequence of stochastic processes* $\{\overline{X}^n,\ n \geq 1\}$ *converge to* $\overline{X} = (M, A, (x_{i,j}), a)$ *in* $\mathcal{W}$ *a.s. and* $\{\mathbb{E}\|M_t^n\|_{L_2}^2, n \geq 1\}$ *is bounded for all* $t \geq 0$. *Then*

(a) the process $X_t := g + M_t + A_t$, $t \geq 0$, takes values in $L_2^{\uparrow}$;
(b) M is a continuous square integrable martingales in L_2 with the quadratic variation

$$\langle\langle M \rangle\rangle_t = \int_0^t \mathrm{pr}_{X_s}\, ds, \tag{4.1}$$

in particular,

$$x_{i,j}(t) = \int_0^t (\mathrm{pr}_{X_s} e_i, e_j)_{L_2} ds, \quad i, j \in \mathbb{N}, \tag{4.2}$$

and

$$a(t) = \int_0^t \| \mathrm{pr}_{X_s} \|_{HS}^2 ds \tag{4.3}$$

for all $t \geq 0$;
(c) $A_t = \int_0^t (\xi - \mathrm{pr}_{X_s} \xi) ds$, $t \geq 0$;
(d) for each $T > 0$, $\int_0^T \| \mathrm{pr}_{X_s^n} - \mathrm{pr}_{X_s} \|_{HS}^2 ds \to 0$ a.s. as $n \to \infty$.

In particular, X is a weak solution to SDE (1.3).

Remark 4.1. *If $\overline{X}^n \to \overline{X} = (M, A, x_{i,j}, a)$ in $\mathcal{W}$ a.s., then M is a continuous square integrable martingale in L_2 with $[M(e_i), M(e_j)] = x_{i,j}$ and $\langle M \rangle = a$. This easily follows from the fact that the weak limit in $C([0,\infty))$ of local martingales is a local martingale (see e.g. Corollary 9.1.19 [22]), the boundedness of $\{\mathbb{E}\|M_t^n\|_{L_2}^2, n \geq 1\}$ and Fatou's lemma.*

Theorem 4.1 will be proved using the deterministic result from Subsection A.3. The following lemmas are needed to check that X_t, $t \in [0,T]$, satisfies the assumptions of Proposition A.2 almost surely.

Lemma 4.1. *Under the assumptions of Theorem 4.1, for each $T > 0$ there exists an random element P^∞ in $L_2([0,T], \mathcal{L}_2(L_2))$ such that*

$$\mathbb{P}\left\{ P^n \to P^\infty \text{ weakly in } L_2([0,T], \mathcal{L}_2(L_2)) \text{ as } n \to \infty \right\} = 1,$$

where $P^n := \mathrm{pr}_{X_\cdot^n}$, $n \geq 1$.

Proof. We set

$$\Omega' := \{\omega : \ \langle M^n \rangle(\omega) \to a(\omega) \text{ in } C([0,T])\} \\ \cap \{\omega : \ [M^n(e_i), M^n(e_j)](\omega) \to x_{i,j}(\omega) \text{ in } C([0,T]) \text{ for all } i, j \in \mathbb{N}\}.$$

It is obvious that $\mathbb{P}\{\Omega'\} = 1$. We take $\omega \in \Omega'$ and show that there exists $P^\infty(\omega) \in L_2([0,T], \mathcal{L}_2(L_2))$ such that $P^n(\omega) \to P^\infty(\omega)$ weakly in $L_2([0,T], \mathcal{L}_2(L_2))$ as $n \to \infty$. Since the sequence

$$\langle M^n \rangle_T(\omega) = \int_0^T \|P_t^n(\omega)\|_{HS}^2 dt = \|P^n(\omega)\|_{T,HS}^2, \quad n \geq 1,$$

converges to $a(T,\omega)$, it is bounded. Thus, by the Banach-Alaoglu theorem, $\{P^n(\omega),\ n \geq 1\}$ is weakly compact in $L_2([0,T], \mathcal{L}_2(L_2))$. Moreover, it has a unique weak limit point denoted by $P^\infty(\omega)$. Indeed, if $\{P^n(\omega),\ n \geq 1\}$ weakly converges to P' along N' and to P'' along N'', then for each $i, j \in \mathbb{N}$ and $t \in [0,T]$

$$\int_0^t (P_s'(\omega)e_i, e_j)_{L_2} ds = \int_0^t (P_s''(\omega)e_i, e_j)_{L_2} ds = x_{i,j}(\omega, t)$$

because

$$(B^t, P^n(\omega))_{T,HS} = \int_0^t (P_s^n(\omega)e_i, e_j)_{L_2} ds \to x_{i,j}(\omega, t) \tag{4.4}$$

for $B_s^t := \mathbb{I}_{[0,t]}(s) e_i \otimes e_j,\ s \in [0,T]$. Hence, Corollary A.1 implies $P' = P''$. Thus, $\{P^n(\omega),\ n \geq 1\}$ weakly converges in $L_2([0,T], \mathcal{L}_2(L_2))$ to $P^\infty(\omega)$.

We note that the measurability of the map $P^\infty : \Omega \to L_2([0,T], \mathcal{L}_2(L_2))$ (here, $P^\infty(\omega) = 0$, if $\omega \notin \Omega'$) will easily follow from the facts that P^∞ is a weak limit of random elements in $L_2([0,T], \mathcal{L}_2(L_2))$ and the Borel σ-algebra on $L_2([0,T], \mathcal{L}_2(L_2))$ coincides with the σ-algebra generated by all continuous linear functionals on $L_2([0,T], \mathcal{L}_2(L_2))$. The lemma is proved. □

Lemma 4.2. *Under the assumptions of Theorem 4.1, for each $T > 0$*

$$\mathbb{P}\Big\{ \|P_t h\|_{L_2} \leq \varliminf_{n\to\infty} \|P_t^n h\|_{L_2},\ \forall t \in [0,T],\ \forall h \in L_2 \Big\} = 1,$$

where $P := \mathrm{pr}_{X_\cdot}$ and $P^n = \mathrm{pr}_{X_\cdot^n}$, $n \geq 1$.

Proof. The lemma immediately follows from the convergence of $\{X_n,\ n \geq 1\}$ a.s. in $C([0,1], L_2^\uparrow)$ and Lemma A.4.

□

Proof of Theorem 4.1. Property (a) easily follows from the closability of $L_2^\uparrow$ in L_2.

Since M is a limit of $(\mathcal{F}_t^{X^n})$-martingales M^n, $n \geq 1$, and $\{\mathbb{E}\|M_t^n\|_{L_2}^2, n \geq 1\}$ is bounded for all $t \geq 0$, one can prove that M also is a square integrable

$(\mathcal{F}_t^X)$-martingale. So, in order to show (b), we only have to check equality (4.2). According to Remark 4.1, (4.1) will follow from (4.2), and equality (4.3) from Lemma 2.1 [19] and (4.1).

By lemmas 4.1, 4.2 and Proposition A.3, trajectories of X and X^n, $n \geq 1$, satisfy conditions $(a)-(c)$ of Proposition A.2 almost surely. Thus,

$$\mathbb{P}\{P^n \to P \text{ weakly in } L_2([0,T], \mathcal{L}_2(L_2)) \text{ as } n \to \infty\} = 1 \tag{4.5}$$

for any fixed $T > 0$. This immediately implies (4.2), by (4.4).

Next, (4.5) and the convergence

$$\|P^n\|^2_{T,HS} = \langle M^n \rangle_T \to a(T) = \langle M \rangle_T = \|P\|^2_{T,HS} \quad \text{a.s. as } n \to \infty$$

yield the strong convergence (d).

For fixed $t \in [0,T]$ and $h \in L_2$ we take

$$B_s := \mathbb{I}_{[0,t]}(s)\xi \otimes h, \quad B_s^n := \mathbb{I}_{[0,t]}(s)\xi_n \otimes h, \quad s \in [0,T], \quad n \geq 1.$$

Then, using the strong convergence of $\{P^n,\ n \geq 1\}$ to P and $\{B^n,\ n \geq 1\}$ to B, we have

$$\int_0^t (\mathrm{pr}_{X_s^n} \xi_n, h)_{L_2} ds = (P^n, B^n)_{T,HS} \to (P, B)_{T,HS} = \int_0^t (\mathrm{pr}_{X_s} \xi, h)_{L_2} ds.$$

Hence, for each $h \in L_2$

$$(A_t, h)_{L_2} = \int_0^t (\xi - \mathrm{pr}_{X_s} \xi, h) ds, \quad t \in [0,T],$$

that implies (c). The theorem is proved. □

4.2. *Proof of Theorem 1.2 (ii).* Let $Y = \{Y(u,t),\ u \in [0,1],\ t \in [0,\infty)\}$ be a random element in $D([0,1], C([0,\infty)))$ and X_t, $t \geq 0$, be a continuous process in $L_2^{\uparrow}$ such that

$$Y(\cdot, t) = X_t \quad \text{in} \quad L_2 \quad \text{a.s.}$$

for all $t \geq 0$.

Remark 4.2. *Since the processes X_t, $t \geq 0$, and $Y(\cdot,t)$, $t \geq 0$, are continuous in L_2, we have*

$$\mathbb{P}\{X_t = Y(\cdot,t) \ \ \textit{in} \ \ L_2 \ \ \textit{for all} \ \ t \geq 0\} = 1.$$

In particular,

$$\mathbb{P}\{\mathrm{pr}_{X_t} = \mathrm{pr}_{Y(\cdot,t)} \ \ \textit{for all} \ \ t \geq 0\} = 1.$$

To prove Theorem 1.2 (ii), we need several auxiliary lemmas.

LEMMA 4.3. *Let $b : [0,1] \times [0,T] \to \mathbb{R}$ be a measurable bounded function such that the function $b(u,t)$, $u \in [0,1]$, is càdlág for all $t \in [0,T]$. Then the function*

$$B(u,t) = \int_0^t b(u,s)ds, \quad u \in [0,1], \quad t \in [0,T],$$

belongs to $D([0,1], C([0,T]))$.

PROOF. Let $u_n \downarrow u$. Then $B(u_n,t) \to B(u,t)$ for any $t \in [0,T]$, by the dominated convergence theorem and the right continuity of $b(\cdot,t)$ for all $t \in [0,T]$. Moreover, by the Arzela-Ascoli theorem, $\{B(u_n,\cdot),\ n \geq 1\}$ is compact in $C([0,T])$. Thus, $B(u_n,\cdot) \to B(u,\cdot)$ in $C([0,T])$. Similarly, $B(u_n,\cdot) \to B(u-,\cdot) := \int_0^{\cdot} b(u-,s)ds$ in $C([0,T])$ as $u_n \uparrow u$. The lemma is proved. $\square$

We define for each $u \in [0,1)$ and $\varepsilon > 0$ the functions from L_2 as follows

$$h_\varepsilon^u(v) = \frac{1}{\varepsilon \wedge (1-u)} \mathbb{I}_{[u,(u+\varepsilon)\wedge 1)}(v), \quad v \in [0,1],$$

and

$$h_\varepsilon^1(v) = \frac{1}{\varepsilon \wedge 1} \mathbb{I}_{[(1-\varepsilon)\vee 0,1]}(v), \quad v \in [0,1].$$

LEMMA 4.4. *Let a function $f(u,t)$, $u \in [0,1]$, $t \in [0,T]$, belong to $D([0,1], C([0,T]))$. Then for each $u \in [0,1]$ the sequence of functions $\{(f(\cdot,t), h_\varepsilon^u)_{L_2},\ t \in [0,T]\}_{\varepsilon>0}$ converges to $f(u,t)$, $t \in [0,T]$, in $C([0,T])$ as $\varepsilon \to 0+$.*

PROOF. We first note that for every $u \in [0,1)$ and $\tilde{\varepsilon} > 0$ there exists $\delta > 0$ such that

$$|f(u,t) - f(v,t)| < \tilde{\varepsilon}, \quad t \in [0,T], \quad v \in [u, u+\delta).$$

In particular, $f(v,t)$, $t \in [0,T]$, $v \in [u,u+\delta)$, is bounded. Hence, for each $\varepsilon \in (0,\delta]$ the function $(f(\cdot,t), h_\varepsilon^u)_{L_2}$, $t \in [0,T]$, belongs to $C[0,T]$, by the dominated convergence theorem, and

$$|(f(\cdot,t), h_\varepsilon^u)_{L_2} - f(u,t)| \leq \int_0^1 |f(v,t) - f(u,t)| h_\varepsilon^u(v)dv < \tilde{\varepsilon}.$$

For $u = 1$ the convergence follows from the same argument and the continuity of $f(v,\cdot)$, $v \in [0,1]$, at $v = 1$. This proves the lemma. $\square$

LEMMA 4.5. *Let $f \in \mathcal{S}^{\uparrow}$ and $0 \le u < v \le 1$. Then*

$$\left(\mathrm{pr}_f\, h^u_\varepsilon, \mathrm{pr}_f\, h^v_\varepsilon\right)_{L_2} \to \frac{1}{m_f(u)} \mathbb{I}_{\{f(u)=f(v)\}} \quad as \quad \varepsilon \to 0+ \tag{4.6}$$

and

$$0 \le \left(\mathrm{pr}_f\, h^u_\varepsilon, \mathrm{pr}_f\, h^v_\varepsilon\right)_{L_2} \le \begin{cases} \frac{1}{v-u-\varepsilon}, & \varepsilon \in (0, v-u)\,, \quad v < 1, \\ \frac{1}{v-u-2\varepsilon}, & \varepsilon \in \left(0, \frac{v-u}{2}\right), \quad v = 1, \end{cases} \tag{4.7}$$

where $m_f(u) = \mathrm{Leb}\{v : \ f(u) = f(v)\}$.

PROOF. Convergence (4.6) follows from Lemma A.1 and a simple calculation.

We show (4.7) only for $v < 1$. So, we fix $\varepsilon \in (0, v-u)$ and consider the following two cases.

a) $f(u+\varepsilon) < f(v)$. Then, by Lemma A.1, $\mathrm{supp}(\mathrm{pr}_f\, h^u_\varepsilon) \cap \mathrm{supp}(\mathrm{pr}_f\, h^v_\varepsilon) = \emptyset$. This implies that

$$\left(\mathrm{pr}_f\, h^u_\varepsilon, \mathrm{pr}_f\, h^v_\varepsilon\right)_{L_2} = 0.$$

b) $f(u+\varepsilon) = f(v)$. Let $\tilde{u}$, $\tilde{v}$ be the ends of the interval $\{r : \ f(v) = f(r)\}$ and $\tilde{u} < \tilde{v}$. Then, $\tilde{u} \le u + \varepsilon < v < \tilde{v}$. Moreover,

$$\left(\mathrm{pr}_f\, h^u_\varepsilon\right)(r) \left(\mathrm{pr}_f\, h^v_\varepsilon\right)(r) = 0, \quad r \notin [\tilde{u}, \tilde{v}),$$

and

$$\begin{aligned} \left(\mathrm{pr}_f\, h^u_\varepsilon\right)(r) \left(\mathrm{pr}_f\, h^v_\varepsilon\right)(r) &= \frac{1}{\varepsilon^2 (\tilde{v} - \tilde{u})^2} \int_{[\tilde{u}, \tilde{v})} \mathbb{I}_{[u, u+\varepsilon)}(r) dr \int_{[\tilde{u}, \tilde{v})} \mathbb{I}_{[v, v+\varepsilon)}(r) dr \\ &\le \frac{1}{(\tilde{v} - \tilde{u})^2}, \quad r \in [\tilde{u}, \tilde{v}). \end{aligned}$$

Hence,

$$\left(\mathrm{pr}_f\, h^u_\varepsilon, \mathrm{pr}_f\, h^v_\varepsilon\right)_{L_2} \le \frac{1}{(\tilde{v} - \tilde{u})^2} \int_0^1 \mathbb{I}_{[\tilde{u}, \tilde{v})}(r) dr \le \frac{1}{v - u - \varepsilon}.$$

The lemma is proved. □

PROOF OF THEOREM 1.2 (II). We first assume that the process X_t, $t \ge 0$, is a weak solution to SDE (1.3) with the martingale part $M := M^X$ and the part of bounded variation $A := A^X$ and check that Y satisfies $(R1) - (R4)$. The idea of proof is similar to the proof of Theorem 6.4 [28]. Namely, we are going to approximate $Y(u, \cdot)$ by $\left\{(X, h^u_\varepsilon)_{L_2}\right\}_{\varepsilon > 0}$.

We first note that property $(R1)$ is trivial.

Let

$$
\begin{aligned}
A^Y(u,t) &:= \int_0^t (\xi(u) - \left(\mathrm{pr}_{Y(\cdot,s)}\, \xi\right)(u))ds \\
&= \int_0^t \left(\xi(u) - \frac{1}{m_Y(u,s)} \int_{\pi_Y(u,s)} \xi(v)dv \right) ds, \quad u \in [0,1], \quad t \geq 0,
\end{aligned}
$$

where the equality follows from Lemma A.1. By Lemma 4.3, $A^Y(u,t)$, $u \in [0,1]$, $t \in [0,T]$, belongs to $D([0,1], C([0,T]))$ for any $T > 0$ and, thus, A^Y belongs to $D([0,1], C([0,\infty)))$. Hence, $M^Y := Y - g - A$ also belongs to $D([0,1], C([0,\infty)))$.

Let h_ε^u be defined as before for each $u \in [0,1]$ and $\varepsilon > 0$. Then, by Lemma 4.4 and Remark 4.2,

$$
(M_\cdot, h_\varepsilon^u)_{L_2} \to M^Y(u,\cdot) \quad \text{in } \; C([0,T]) \; \text{ a.s. } \; \text{as } \; \varepsilon \to 0+ \tag{4.8}
$$

and

$$
(X_\cdot, h_\varepsilon^u)_{L_2} \to Y(u,\cdot) \quad \text{in } \; C([0,T]) \; \text{ a.s. } \; \text{as } \; \varepsilon \to 0+ \tag{4.9}
$$

for all $T > 0$. Thus, Y satisfies $(R2)$, by Proposition A.1 [28] and Remark 4.2. We also note that (4.9) yields $\mathcal{F}_t^Y = \mathcal{F}_t^X$ for all $t \geq 0$.

Taking arbitrary $u \in (0,1)$, $\varepsilon_0 \in (0, u \wedge (1-u))$ and using Proposition A.1 [28], we have

$$
|(X_t, h_\varepsilon^u)_{L_2}| \leq |(X_t, h_{\varepsilon_0}^0)_{L_2}| + |(X_t, h_{\varepsilon_0}^1)_{L_2}|
$$

for all $t \geq 0$ and $\varepsilon \in (0, \varepsilon_0]$. Hence,

$$
\begin{aligned}
|(M_t, h_\varepsilon^u)_{L_2}| &\leq |(X_t, h_{\varepsilon_0}^0)_{L_2}| + |(X_t, h_{\varepsilon_0}^1)_{L_2}| + 2t\|\xi\|_{L_\infty} \\
&\leq |(M_t, h_{\varepsilon_0}^0)_{L_2}| + |(M_t, h_{\varepsilon_0}^1)_{L_2}| + 6t\|\xi\|_{L_\infty}
\end{aligned}
$$

for all $t \geq 0$ and $\varepsilon \in (0, \varepsilon_0]$. Here, the estimate $\|\mathrm{pr}_{X_t} \xi\|_{L_\infty} \leq \|\xi\|_{L_\infty}$ follows from Lemma A.1. Using (4.8), the fact that $(M_t, h_\varepsilon^u)_{L_2}$, $t \geq 0$, is a square integrable $(\mathcal{F}_t^Y)$-martingale for every $\varepsilon > 0$ and the dominated convergence theorem, we have that $M^Y(u,t)$, $t \geq 0$, is a square integrable $(\mathcal{F}_t^Y)$-martingale. In the case $u \in \{0,1\}$, one can use the convergence of $\{(M_\cdot, h_\varepsilon^u)\}_{\varepsilon>0}$ to $M^Y(u,\cdot)$ and $\{(X_\cdot, h_\varepsilon^u)\}_{\varepsilon>0}$ to $Y(u,\cdot)$ in $C[0,T]$ a.s. and the localisation sequence of $(\mathcal{F}_t^Y)$-stopping times in order to prove that $M^Y(u,t)$, $t \geq 0$, is a local $(\mathcal{F}_t^Y)$-martingale. This proves $(R3)$.

Next, (4.8), Lemma B.11 [6] and the polarization formula for joint quadratic variation of martingales yield

$$\left[(M_\cdot, h_\varepsilon^u)_{L_2}, (M_\cdot, h_\varepsilon^v)_{L_2}\right] \to \left[M^Y(u,\cdot), M^Y(v,\cdot)\right]$$

in $C([0,\infty))$ in probability as $\varepsilon \to 0+$ for all $u, v \in [0,1]$.

By the finiteness of the expectation $\mathbb{E}\|M_t\|_{L_2}^2 < \infty$, the equality

$$\mathbb{E}\int_0^t \|\operatorname{pr}_{Y(\cdot,s)}\|_{HS}^2 ds = \mathbb{E}\int_0^t \|\operatorname{pr}_{X_s}\|_{HS}^2 ds = \mathbb{E}\|M_t\|_{L_2}^2 < \infty, \quad t \geq 0,$$

and Lemma A.3, we have

(4.10)
$$\mathbb{P}\left\{\exists R \subseteq [0,\infty) \text{ s.t. } \operatorname{Leb}([0,\infty)\setminus R) = 0 \text{ and } Y(\cdot,t) \in \mathcal{S}^\uparrow \ \forall t \in R\right\} = 1.$$

Thus, applying Lemma 4.5 to $f = Y(\cdot,t,\omega)$ and using the dominated convergence theorem, we obtain

$$\begin{aligned}\left[(M_\cdot, h_\varepsilon^u)_{L_2}, (M_\cdot, h_\varepsilon^v)_{L_2}\right]_t &= \int_0^t \left(\operatorname{pr}_{Y(\cdot,s)} h_\varepsilon^u, \operatorname{pr}_{Y(\cdot,s)} h_\varepsilon^v\right)_{L_2} ds \\ &\to \int_0^t \frac{\mathbb{I}_{\{Y(u,s)=Y(v,s)\}}}{m_Y(u,s)} ds \quad \text{a.s. as } \varepsilon \to 0+\end{aligned}$$

for any $t \geq 0$. This implies $(R4)$ for all $u, v \in [0,1]$, $u \neq v$.

So, to finish the proof of the theorem, we have to check $(R4)$ for $u = v \in [0,1]$. Since $M^Y \in D([0,1], C([0,\infty)))$, we have

$$M^Y(v,\cdot) \to M^Y(u,\cdot) \quad \text{in } C([0,\infty)) \text{ a.s.}$$

as $v \downarrow u$, if $u < 1$, and $v \uparrow u$, if $u = 1$. Thus, by Lemma B.11 [6] and the polarization formula for joint quadratic variation of martingales,

$$\left[M^Y(v,\cdot), M^Y(u,\cdot)\right] \to \left[M^Y(u,\cdot)\right] \quad \text{in } C([0,\infty)) \text{ in probability.}$$

Using (4.10), Lemma A.1 and the monotone convergence theorem, it is easily seen that for each $t \geq 0$

$$\left[M^Y(v,\cdot), M^Y(u,\cdot)\right]_t = \int_0^t \frac{\mathbb{I}_{\{Y(u,s)=Y(v,s)\}}}{m_Y(u,s)} ds \to \int_0^t \frac{ds}{m_Y(u,s)}$$

as $v \downarrow u$, if $u < 1$, and $v \uparrow u$, if $u = 1$. The firs part of theorem is proved.

The inverse statement, where Y satisfies $(R1) - (R4)$ follows from trivial computations and Remark 1.4 (ii).

□

5. Proof of theorems 1.1 and 1.2. In this section, we finishe the proof of the main result of the paper.

PROOF OF THEOREM 1.2 (I). We recall that, by propositions 2.2 and 2.3, for every $\xi \in \mathcal{S}^\uparrow$ and $g \in \Theta_\xi$ there exists a weak solution to SDE (1.3) satisfying $(R1)-(R4)$, where Θ_ξ is defined in Proposition 2.2. Moreover, Θ_ξ is dense in $L_2^\uparrow(\xi)$, since $\Xi^\xi(L_2^\uparrow(\xi) \setminus \Theta_\xi) = 0$ and $\operatorname{supp} \Xi^\xi = L_2^\uparrow(\xi)$ (see Proposition 2.1).

In order to prove the statement, we first construct sequences $\{g_n,\ n \geq 1\} \in \mathcal{S}^\uparrow$ and $\{\xi_n,\ n \geq 1\} \in \mathcal{S}^\uparrow$ such that $g_n \in \Theta_{\xi_n}$ for all $n \geq 1$, $g_n \to g$, $\xi_n \to \xi$ in $L_{2+\delta}$ and $\{\xi_n(1) - \xi_n(0),\ n \geq 1\}$ bounded. We set

$$\xi_n := \sum_{k=1}^{2^n} \left(\frac{k}{2^{2n}} + \xi\left(\frac{k-1}{2^n}\right)\right) \mathbb{I}_{\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)} + \left(\frac{1}{2^n} + \xi(1)\right) \mathbb{I}_{\{1\}}, \quad n \geq 1.$$

Since ξ is discontinuous at most in a countable number of points, $\xi_n \to \xi$ a.e. and, thus, it convergences in $L_{2+\delta}$, by the dominated convergence theorem. Moreover, $\xi_n(1) - \xi_n(0) = \xi(1) - \xi(0)$ for all $n \geq 1$. We also note that

$$L_2^\uparrow(\xi_n) = \left\{ f \in L_2^\uparrow :\ f \text{ is } \sigma^*\left(\left\{\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right),\ k \in [2^n]\right\}\right)\text{-measurable}\right\},$$

due to the term $\frac{k}{2^{2n}}$ in the definition of ξ_n and the monotonicity of ξ. To construct g_n, $n \geq 1$, we first set

$$\tilde{g}_n := \operatorname{pr}_{\xi_n} g = \sum_{k=1}^{n} y_k^n \mathbb{I}_{\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)}, \quad n \geq 1,$$

where

$$y_k^n = 2^n \int_{\frac{k-1}{2^n}}^{\frac{k}{2^n}} g(v)dv.$$

Since $\tilde{g}_n \in L_2^\uparrow(\xi_n)$, the set Θ_{ξ_n} is dense in $L_2^\uparrow(\xi_n)$ and $\|\cdot\|_{L_{2+\delta}}$ is equivalent to $\|\cdot\|_{L_2}$ in $L_2(\xi_n)$ because $L_2(\xi_n)$ is finite dimensional, for each $n \geq 1$ we can find $g_n \in \Theta_{\xi_n}$ satisfying $\|g_n - \tilde{g}_n\|_{L_{2+\delta}} < \frac{1}{n}$. Using e.g. Theorem 1 [1] and Remark A.1 (ii), we obtain $g_n \to g$ in $L_{2+\delta}$.

Let X^n be a weak solution to SDE (1.3) with g and ξ replaced by g_n and ξ_n, which exists according to Proposition 2.2. Let $\overline{X}^n$, $n \geq 1$, be defined by (3.1). Then, by Corollary 3.1, there exists a subsequence $N \subseteq \mathbb{N}$ such that $\overline{X}^n \to \overline{X}$ in $\mathcal{W}$ in distribution along N. Next, by the Skorohod representation theorem (see Theorem 3.1.8 [16]), we can found a probability space and define there random elements $\overline{Z}$, $\overline{Z}^n$, $n \in N$, taking values in $\mathcal{W}$ such that $\mathrm{Law}(\overline{X}) = \mathrm{Law}(\overline{Z})$, $\mathrm{Law}(\overline{X}^n) = \mathrm{Law}(\overline{Z}^n)$ and $\overline{Z}^n \to \overline{Z}$ in $\mathcal{W}$ a.s. along N. We also note that $\{\mathbb{E}\|M_t^{Z^n}\|_{L_2}^2 = \mathbb{E}\|M_t^{X^n}\|_{L_2}^2,\ n \geq 1\}$ is bounded for all $t \geq 0$, by Corollary 2.5 and Remark 1.4 (iii). Thus, the existence of sulutions ot equation (1.3) follows from Theorem 4.1. □

Proof of Theorem 1.1. In order to prove Theorem 1.1, we will check that there exists a solution to the equation (1.3) which has a modification from the Skorohod space $D([0,1], C[0,\infty))$. The approach will be similar as in the proof of Theorem 1.2 (ii), but now we have to construct sequences $\{g_n,\ n \geq 1\} \in \mathcal{S}^\uparrow$ and $\{\xi_n,\ n \geq 1\} \in \mathcal{S}^\uparrow$ which also satisfy conditions $(c1)$, $(c2)$ of Proposition 3.2. It is possible to do due to the additional assumptions on g and ξ in the theorem.

Letting $u_{i,j}^n$, $j = 0, \dots, 2^n$, be the uniform partition of $[\tilde{u}_{i-1}, \tilde{u}_i]$ for each $i \in [l]$ and $n \geq 1$, we define the functions $\iota_n : [0,1] \to [0,1]$ as follows

$$\iota_n(u) = \sum_{i=1}^{l} \sum_{j=1}^{2^n} u_{i,j-1}^n \mathbb{I}_{[u_{i,j-1}^n, u_{i,j}^n)}(u) + u_{l,n-1}^n \mathbb{I}_{\{u_{l,n}^n\}}(u), \quad u \in [0,1], \quad n \geq 1.$$

Then the functions ι_n belong to $D^\uparrow$ and map $[\tilde{u}_{i-1}, \tilde{u}_i)$ into $[\tilde{u}_{i-1}, \tilde{u}_i)$ for any $i \in [l]$. We put for every $n \geq 1$

$$\xi_n := \left(\xi + \frac{1}{n}\mathrm{id}\right) \circ \iota_n{}^{11} \quad \text{and} \quad \tilde{g}_n := g \circ \iota_n,$$

where id denotes the identity function on $[0,1]$. Then similarly as before, for each $n \geq 1$ we can take $g_n \in \Theta_{\xi_n}$ such that $\|g_n - \tilde{g}_n\|_{L_\infty} < \frac{1}{2^{\gamma n}}$. By the boundedness of g and ξ and the dominated convergence theorem, $g_n \to g$ and $\xi_n \to \xi$ in $L_{2+\delta}$. Moreover, $\{\xi_n(1) - \xi_n(0),\ n \geq 1\}$ is bounded.

Next, we check $(c1)$ and $(c2)$ for every $\pi = [\tilde{u}_{i-1}, \tilde{u}_i]$, $i \in [l]$. We recall that $g, \xi \in D^\uparrow$ are γ-Hölder continuous on each interval $(\tilde{u}_{i-1}, \tilde{u}_i)$ with $\gamma > \frac{1}{2}$. So, there exists $C > 0$ such that

$$|g(u) - g(v)| \vee |\xi(u) - \xi(v)| \leq C|u - v|^\gamma, \quad u, v \in (\tilde{u}_{i-1}, \tilde{u}_i), \quad i \in [l].$$

Let $i \in [l]$ be fixed and $(a,b) := (\tilde{u}_{i-1}, \tilde{u}_i)$. We take $u \in (a,b)$ and $r > 0$ such that $u+r, u-r \in (a,b)$ and estimate $[g_n(u+r) - g_n(u)][g_n(u) - g_n(u-r)]$ for each $n \geq 1$. First, we note that for $r < \frac{b-a}{2^{n+1}}$

$$[g_n(u+r) - g_n(u)][g_n(u) - g_n(u-r)] = 0,$$

[11]The fanction $\frac{1}{n}\mathrm{id}$ is needed here in order to have $\sigma^*(\xi_n) = \sigma^*\left([u_{i,j-1}, u_{i,j})\,,\ j \in [2^n],\ i \in [l]\right)$.

since $g_n(u+r) - g_n(u) = 0$ or $g_n(u) - g_n(u-r) = 0$. Next, let $r \geq \frac{b-a}{2^{n+1}}$. Then

$$\begin{aligned}
&[g_n(u+r) - g_n(u)][g_n(u) - g_n(u-r)] \\
&\qquad \leq \left[\tilde{g}_n(u+r) - \tilde{g}_n(u) + \frac{2}{2^{\gamma n}}\right]\left[\tilde{g}_n(u) - \tilde{g}_n(u-r) + \frac{2}{2^{\gamma n}}\right] \\
&\qquad = \left[g(\iota_n(u+r)) - g(\iota_n(u)) + \frac{2}{2^{\gamma n}}\right]\left[g(\iota_n(u)) - g(\iota_n(u-r)) + \frac{2}{2^{\gamma n}}\right] \\
&\qquad \leq \left[C(\iota_n(u+r) - \iota_n(u))^\gamma + \frac{2}{2^{\gamma n}}\right]\left[C(\iota_n(u) - \iota_n(u-r))^\gamma + \frac{2}{2^{\gamma n}}\right] \\
&\qquad \leq \left(3^\gamma C + \frac{2^{\gamma+1}}{(b-a)^\gamma}\right)^2 r^{2\gamma},
\end{aligned}$$

since

$$[\iota_n(u+r) - \iota_n(u)] \vee [\iota_n(u) - \iota_n(u-r)] \leq r + \frac{b-a}{2^n} \leq 3r \quad \text{for } r \geq \frac{b-a}{2^{n+1}}.$$

Similarly,

$$\begin{aligned}
&[g_n(u+r) - g_n(u)][\xi_n(u) - \xi_n(u-r)] \leq \tilde{C} r^{2\gamma}, \\
&[\xi_n(u+r) - \xi_n(u)][g_n(u) - g_n(u-r)] \leq \tilde{C} r^{2\gamma}, \\
&[\xi_n(u+r) - \xi_n(u)][\xi_n(u) - \xi_n(u-r)] \leq \tilde{C} r^{2\gamma}.
\end{aligned}$$

Thus, $\{g_n,\ n \geq 1\}$ and $\{\xi_n,\ n \geq 1\}$ satisfy $(c1)$ of Proposition 3.2 with $\beta = 2\gamma > 1$ and $\pi = [\tilde{u}_{i-1}, \tilde{u}_i]$.

Estimate $(c2)$ can be proved similarly, using the Hölder continuity of g and ξ on $(\tilde{u}_{i-1}, \tilde{u}_i)$ and the form of the maps ι_n, $n \geq 1$.

As before, let X_t^n, $t \geq 0$, be a weak solution to SDE (1.3) with g and ξ replaced by g_n and ξ_n. Let $\overline{X}^n$, $n \geq 1$, be defined by (3.1). Let also $\{Y^n(u,t),\ u \in [0,1],\ t \geq 0\}$, $n \geq 1$, be random elements in $D([0,1], C[0,\infty))$ satisfying $(R1) - (R4)$ and $X_t^n = Y^n(\cdot, t)$ in L_2 for all $t \geq 0$ almost surely. Such random elements exists by Proposition 2.3.

By propositions 3.1, 3.2, A.4 and Remark 3.1, the sequence $\{(\overline{X}^n, Y^n),\ n \geq 1\}$ is tight in $\mathcal{W} \times D([0,1], C([0,\infty)))$. As before, we can fine a subsequence $N \subseteq \mathbb{N}$, a probability space and a family of random elements $\{(\overline{Z}, \overline{V}),\ (\overline{Z}^n, \overline{V}^n),\ n \in N\}$ on this probability space such that $\mathrm{Law}(\overline{X}^n, Y^n) = \mathrm{Law}(\overline{Z}^n, V^n)$, $\mathrm{Law}(\overline{X}, Y) = \mathrm{Law}(\overline{Z}, V)$ and $(\overline{Z}^n, V^n) \to (\overline{Z}, V)$ in $\mathcal{W} \times D([0,1], C([0,\infty)))$ a.s. along N. Then, by Proposition 4.1 and Corollary 2.5, the process $Z := g + A^Z + M^Z$ is a weak solution to SDE (1.3), where $\overline{Z} = (M^Z, A^Z, (x_{i,j}^Z), a^Z)$. Moreover, $Z_t = V(\cdot, t)$ in L_2 for all $t \geq 0$ almost surely. Hence, V satisfies $(R1)$-$(R4)$, by Theorem 1.2 (ii). The theorem is proved. $\square$

APPENDIX A: APPENDIX

A.1. The sitting time at zero of non-negative semimartingales.

PROPOSITION A.1. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a complete right-continuous filtration and $y(t)$, $t \geq 0$, be a continuous non-negative $(\mathcal{F}_t)$-semimartingale such that*

$$y(t) = y_0 + \int_0^t \rho(s)\mathbb{I}_{\{y(s)>0\}}dB(s) + \xi_0 \int_0^t \mathbb{I}_{\{y(s)=0\}}ds, \tag{A.1}$$

where y_0, ξ_0 are non-negative constants, $\rho(t)$, $t \geq 0$, is an $(\mathcal{F}_t)$-predictable process taking values in $[1, C]$, for some (non-random) constant C, and $B(t)$, $t \geq 0$, is an $(\mathcal{F}_t)$-Brownian motion. Then

$$\mathbb{E}\int_0^t \mathbb{I}_{\{y(s)>0\}}ds \leq \sqrt{\frac{2t}{\pi}}(\xi_0 t + y_0), \quad t \geq 0.$$

PROOF. We set

$$R_t := \int_0^t \mathbb{I}_{\{y(s)>0\}}ds, \quad t \geq 0,$$

and use the idea from [15, P. 998-999] in order to estimate $\mathbb{E}R_t$. We will consider two cases.

Case I: $\xi_0 > 0$.

To estimate $\mathbb{E}R_t$, we first show that $R_t \uparrow \infty$ a.s. as $t \uparrow \infty$. Let us note that

$$N_t^\rho := \int_0^t \rho(s)\mathbb{I}_{\{y(s)>0\}}dB(s), \quad t \geq 0,$$

is a continuous martingale with the quadratic variation

$$[N^\rho]_t = \int_0^t \rho(s)^2\mathbb{I}_{\{y(s)>0\}}ds, \quad t \geq 0,$$

that increases to a random variable R_∞^ρ, taken values in $[0, \infty]$, as t increases to infinity. Consequently, $N_t^\rho \to N_\infty^\rho$ in $\mathbb{R}$ a.s. on $\{R_\infty^\rho < \infty\}$. Let $R_t \uparrow R_\infty$ as $t \uparrow \infty$. Since $\rho(t) \leq C$ for all $t \geq 0$, we have that $\{R_\infty < \infty\} \subseteq \{R_\infty^\rho < \infty\}$ and, thus, $N_t^\rho \to N_\infty^\rho$ a.s. on $\{R_\infty < \infty\}$. Setting $R_t^0 := \int_0^t \mathbb{I}_{\{y(s)=0\}}ds$ and noting that $R_t + R_t^0 = t$, $t \geq 0$, we see that $R_t^0 \uparrow \infty$ on $\{R_\infty < \infty\}$ as $t \uparrow \infty$. But then, $y(t) = y_0 + N_t^\rho + \xi_0 R_t^0 \to \infty$ a.s. on $\{R_\infty < \infty\}$ as $t \to \infty$, by (A.1). This contradicts the fact that $R_t^0 \uparrow \infty$ on $\{R_\infty < \infty\}$ unless its probability is zero. So, we have showed that $R_\infty = \infty$ a.s.

Since $R_t \uparrow \infty$ as $t \uparrow \infty$, it follows that its (right) inverse $t \mapsto A_t$, defined by

$$A_t := \inf\{s \geq 0 : \ R_s > t\},$$

is finite for all $t \geq 0$. Note that A_t, $t \geq 0$, is increasing and right-continuous. Moreover, A_t is an $(\mathcal{F}_t)$-stopping time for all $t \geq 0$.

Next, we set

$$N'_t := N^\rho_{A_t} = \int_0^{A_t} \rho(s)\mathbb{I}_{\{y(s)>0\}} dB(s), \quad t \geq 0.$$

Since N^ρ is an $(\mathcal{F}_t)$-martingale, N' is an $(\mathcal{F}'_t)$-martingale, where $\mathcal{F}'_t := \mathcal{F}_{A_t}$. Moreover, N' is continuous because R_t, $t \geq 0$, is constant on each $[A_{s-}, A_s]$ and, therefore, $y(t)$, $t \geq 0$, equals zero almost everywhere on $[A_{s-}, A_s]$.

Denoting

$$Q_t := \int_0^{R_t} \rho(A_s)^2 ds, \quad t \geq 0,$$

and using the change of variables formula, we can see that

$$Q_t = \int_0^t \rho(A_{R_s})^2 dR_s = \int_0^t \rho(A_{R_s})^2 \mathbb{I}_{\{y(s)>0\}} ds = \int_0^t \rho(s)^2 \mathbb{I}_{\{y(s)>0\}} ds$$

for all $t \geq 0$, since $A_{R_s} = \max\{r : \ R_r = R_s\} = s$ if $y(s) > 0$. Thus,

$$\begin{aligned} [N']_t = [N^\rho]_{A_t} &= \int_0^{A_t} \rho(s)^2 \mathbb{I}_{\{y(s)>0\}} ds \\ &= Q_{A_t} = \int_0^{R_{A_t}} \rho(A_s)^2 ds = \int_0^t \rho(A_s)^2 ds \end{aligned} \tag{A.2}$$

for any $t \geq 0$.

Next, by the Tanaka formula

$$\begin{aligned} y(t) = |y(t)| = y_0 &+ \int_0^t \operatorname{sgn} y(s) dy(s) + l_t(y) \\ &= y_0 + \int_0^t \rho(s)\mathbb{I}_{\{y(s)>0\}} dB(s) + l_t(y) \\ &= y_0 + N^\rho_t + l_t(y), \quad t \geq 0, \end{aligned}$$

where $l_t(y)$ denotes the local time of y at zero and equals

$$l_t(y) = \left\{ -y_0 - \inf_{s \leq t} \int_0^s \operatorname{sgn} y(r) dy(r) \right\} \vee 0 = \left\{ -y_0 - \inf_{s \leq t} N^\rho_s \right\} \vee 0, \quad t \geq 0,$$

by Theorem 22.1 [24]. Hence, by (A.1),

$$l_t(y) = \xi_0 \int_0^t \mathbb{I}_{\{y(s)=0\}} ds, \quad t \geq 0.$$

Consequently,

$$
\begin{aligned}
t = R_{A_t} &= \int_0^{A_t} \mathbb{I}_{\{y(s)>0\}} ds = A_t - \int_0^{A_t} \mathbb{I}_{\{y(s)=0\}} ds \\
&= A_t - \frac{1}{\xi_0} l_{A_t}(y) = A_t - \frac{1}{\xi_0}\left\{-y_0 - \inf_{s\le A_t} N_s^\rho\right\} \vee 0 \\
&= A_t - \frac{1}{\xi_0}\left\{-y_0 - \inf_{s\le t} N_{A_s}^\rho\right\} \vee 0 = A_t - \frac{1}{\xi_0}\left\{-y_0 - \inf_{s\le t} N_s'\right\} \vee 0.
\end{aligned}
$$

So, $t = A_t - \frac{1}{\xi_0}\{-y_0 - \inf_{s\le t} N_s'\} \vee 0$, $t \ge 0$, which implies that A_t, $t \ge 0$, is strictly increasing and continuous. Thus,

$$
\begin{aligned}
R_t = \max\{s :\ A_s \le t\} &= \max\left\{s :\ s + \frac{1}{\xi_0}\left[-y_0 - \inf_{r\le s} N_r'\right] \vee 0 \le t\right\} \\
&= \max\left\{s :\ \left[-y_0 - \inf_{r\le s} N_r'\right] \vee 0 \le \xi_0(t-s)\right\} \\
&= \max\left\{s :\ -y_0 - \inf_{r\le s} N_r' \le \xi_0(t-s),\ s \le t\right\} \\
&\le \max\left\{s :\ \sup_{r\le s}(-N_r') \le \xi_0 t + y_0\right\} \wedge t = \tau_{\xi_0 t + y_0}^{N'} \wedge t,
\end{aligned}
$$

where $\tau_a^{N'} := \inf\{t :\ -N_t' = a\}$. Denoting $\sigma_a := \inf\{t :\ B(t) = a\}$ and using the inequality $[N']_t \ge t$, $t \ge 0$, (see (A.2)) and Lemma 2.4 [27], we obtain

$$
\mathbb{E}R_t \le \mathbb{E}(\tau_{\xi_0 t+y_0}^{N'} \wedge t) \le \mathbb{E}(\sigma_{\xi_0 t + y_0} \wedge t) \le \sqrt{\frac{2t}{\pi}}(\xi_0 t + y_0).
$$

Case II: $\xi_0 = 0$.

In this case, $y(t) = y_0 + \int_0^t \rho(s)\mathbb{I}_{\{y(s)>0\}} dB(s)$, $t \ge 0$, is a continuous positive martingale. It implies that y stays at zero for all $t \ge \tau_{y_0}^y := \inf\left\{t :\ -\int_0^t \rho(s)\mathbb{I}_{\{y(s)>0\}} dB(s) = y_0\right\}$. Hence, using Lemma 2.4 [27] again and the fact that $\int_0^t \rho(s)\mathbb{I}_{\{y(s)>0\}} dB(s) = \int_0^t \rho(s) dB(s)$ for all $t \in [0, \tau_{y_0}^y]$, we have

$$
\mathbb{E}R_t = \mathbb{E}\tau_{y_0}^y \le \mathbb{E}(\sigma_{y_0} \wedge t) \le \sqrt{\frac{2t}{\pi}} y_0.
$$

Combining these two cases, we obtain the estimate

$$
\mathbb{E}R_t \le \sqrt{\frac{2t}{\pi}}(\xi_0 t + y_0), \quad t \ge 0.
$$

The proposition is proved. □

A.2. The projection operator. We recall that for $g \in L_2^{\uparrow}$ the projection operator in L_2 on the closed linear subspace

$$L_2(g) = \{f \in L_2 : \ f \text{ is } \sigma^*(g)\text{-measurable}\}$$

is denoted by pr_g.

REMARK A.1. *(i) The operator* pr_g *is well-defined, since for two functions g_1 and g_2 coinciding a.e., $\sigma^*(g_1) = \sigma^*(g_2)$.*

(ii) For each $h \in L_2$, $\mathrm{pr}_g h$ *coincides a.e. with the conditional expectation $\mathbb{E}(h|\sigma(g))$ on the probability space* $([0,1], \mathcal{B}([0,1]), \mathrm{Leb})$, *where* $\mathcal{B}([0,1])$ *denotes the Borel σ-algebra on* $[0,1]$.

For fixed $g \in D^{\uparrow}$ we will denote the family of intervals $I(c) = g^{-1}(\{c\}) = \{u : \ g(u) = c\}$, $c \in \mathbb{R}$, satisfying $\mathrm{Leb}(I(c)) > 0$ by $\mathcal{K}_g$. We note that either $I_1 \cap I_2 = \emptyset$ or $I_1 = I_2$ for any $I_1, I_2 \in \mathcal{K}_g$. This implies that $\mathcal{K}_g$ is countable. Let

$$G_g := \bigcup_{I \in \mathcal{K}_g} I \quad \text{and} \quad F_g := (0,1) \setminus G_g.$$

For any function $h \in L_2$ we define the function

$$h_g(u) := \begin{cases} \frac{1}{\mathrm{Leb}(I)} \int_I h(v)dv, & u \in I \in \mathcal{K}_g, \\ h(u), & u \in F_g, \end{cases} \qquad u \in (0,1). \tag{A.3}$$

LEMMA A.1. *Let $g \in D^{\uparrow}$ and $h \in L_2$. Then* $\mathrm{pr}_g h = h_g$ *a.e.*

PROOF. In order to prove the lemma, we first show that there exists a Borel function $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$h_g = \varphi(g).$$

This will imply the measurability of h_g with respect to $\sigma^*(g)$.

Since g is a non-decreasing function, the restriction $g|_{F_g}$ of g to the Borel set F_g is an one-to-one map from F_g to $g(F_g) = \{g(u) : \ u \in F_g\}$. By Kuratowski's theorem (see Theorem A.10.5 [16]), $g(F_g)$ is a Borel subset of $\mathbb{R}$ and $\left(g|_{F_g}\right)^{-1}$ is a Borel measurable function from $g(F_g)$ to F_g. Thus, we define

$$\varphi(x) = h\left(\left(g|_{F_g}\right)^{-1}(x)\right), \quad x \in g(F_g).$$

If $x \in g((0,1)) \setminus g(F_g)$, then there exists an unique interval $I_x \in \mathcal{K}_g$ such that $g(u) = x$ for all $u \in I_x$. Hence, we can define

$$\varphi(x) = \begin{cases} \frac{1}{\mathrm{Leb}(I_x)} \int_{I_x} h(v)dv, & x \in g((0,1)) \setminus g(F_g), \\ 0, & x \notin g((0,1)). \end{cases}$$

By the construction of φ, it is easy to see that φ is a Borel function and for all $u \in (0,1)$

$$\varphi(g(u)) = h_g(u).$$

Next, taking an arbitrary $\sigma^*(g)$-measurable function $f \in L_2$ and noting that there exists a Borel function $\psi : \mathbb{R} \to \mathbb{R}$ such that $f = \psi(g)$ a.e., we can estimate the norm $\|f - h\|_{L_2}^2$. So,

$$\begin{aligned}\int_0^1 (f(u) - h(u))^2 du = \int_0^1 (\psi(g(u)) - h(u))^2 du &\geq \sum_{I \in \mathcal{K}_g} \int_I (\psi(c_I) - h(u))^2 du \\ &\geq \sum_{I \in \mathcal{K}_g} \int_I \left(\frac{1}{\mathrm{Leb}(I)} \int_I h(v) dv - h(u) \right)^2 du \\ &= \int_0^1 (h_g(u) - h(u))^2 du,\end{aligned}$$

where $c_I = g(u)$, $u \in I$, and the last inequality is obtained by minimising of the map

$$\theta \mapsto \int_I (\theta - h(u))^2 du.$$

This finishes the proof of the lemma. $\square$

Lemma A.2. *For each $g \in D^{\uparrow}$ the projection operator pr_g maps $L_2^{\uparrow}$ into $L_2^{\uparrow}$.*

The statement easily follows from the explicit formula (A.3) for $\mathrm{pr}_g h$.

Let $g : (0,1) \to \mathbb{R}$ be a non-decreasing function. We define

$$m_g(u) := \mathrm{Leb}\{v : \ g(u) = g(v)\}, \quad u \in (0,1). \tag{A.4}$$

Remark A.2. *If $g_1 = g_2$ a.e., then $m_{g_1} = m_{g_2}$ a.e. Thus, m_g is well-defined for any $g \in L_2^{\uparrow}$.*

Lemma A.3. *Let $g \in L_2^{\uparrow}$ and m_g be defined by (A.4). Then*

$$\| \mathrm{pr}_g \|_{HS}^2 = \int_0^1 \frac{du}{m_g(u)} = \# g.$$

In particular, $\| \mathrm{pr}_g \|_{HS}^2 < \infty$ if and only if the càdlág modification of g belongs to $\mathcal{S}^{\uparrow}$.

PROOF. We take $\tilde{g} \in D^{\uparrow}$ such that $g = \tilde{g}$ a.e. and note that $\|\operatorname{pr}_g\|_{HS}^2 = \#g$ follows from Lemma 6.1 [28]. Moreover, $\|\operatorname{pr}_g\|_{HS}^2 < \infty$ if and only if $\tilde{g} \in \mathcal{S}^{\uparrow}$. So, we only have to show that $\int_0^1 \frac{du}{m_g(u)} = \#g$.

Let $\mathcal{K}_{\tilde{g}}$ and $F_{\tilde{g}}$ be defined as in the beginning of the present section. Then, obviously, $m_{\tilde{g}}(u) = 0$ if and only if $u \in F_{\tilde{g}}$.

If $\int_0^1 \frac{du}{m_g(u)} < \infty$, then $\operatorname{Leb}(F_{\tilde{g}}) = 0$ and

$$\int_0^1 \frac{du}{m_g(u)} = \sum_{I \in \mathcal{K}_{\tilde{g}}} \int_I \frac{du}{m_g(u)} = \#\mathcal{K}_{\tilde{g}}, \tag{A.5}$$

where $\#\mathcal{K}_{\tilde{g}}$ denotes the number of distinct intervals in $\mathcal{K}_{\tilde{g}}$. Since $\tilde{g}$ is càdlág, $\#\mathcal{K}_{\tilde{g}} < \infty$ and $\operatorname{Leb}(F_{\tilde{g}}) = 0$, one can see that $F_{\tilde{g}} = \emptyset$. This implies that $\#\tilde{g} = \#\mathcal{K}_{\tilde{g}}$. Thus, $\#\tilde{g} \leq \int_0^1 \frac{du}{m_g(u)}$ (including the trivial case $\int_0^1 \frac{du}{m_g(u)} = +\infty$).

Next, if $\#\tilde{g} < \infty$, then $\tilde{g} \in \mathcal{S}^{\uparrow}$, and, consequently, $F_{\tilde{g}} = \emptyset$. This together with (A.5) yield $\int_0^1 \frac{du}{m_g(u)} \leq \#\tilde{g}$. The lemma is proved. $\square$

LEMMA A.4. *For each $h \in L_2$ the map $g \mapsto \|\operatorname{pr}_g h\|_{L_2}$ from $L_2^{\uparrow}$ to $\mathbb{R}$ is lower semi-continuous, that is,*

$$\|\operatorname{pr}_g h\|_{L_2} \leq \varliminf_{n\to\infty} \|\operatorname{pr}_{g_n} h\|_{L_2},$$

for each sequence $\{g_n,\ n \geq 1\}$ converging to g in $L_2^{\uparrow}$.

PROOF. We first note that it is enough to prove the lemma only for $g_n \to g =: g_0$ a.e., since every convergent sequence in L_2 contains an convergent a.e. subsequence.

Let

$$J := \left\{x \in \mathbb{R} :\ \operatorname{Leb}(g_n^{-1}(\{x\})) = 0 \text{ for all } \ n \geq 0\right\}.$$

Then $\operatorname{Leb}(\mathbb{R} \setminus J) = 0$, due to the countability of $\mathbb{R} \setminus J$. Thus, J is dense in $\mathbb{R}$ and, consequently, we can choose an increasing sequence of finite subsets $J_k \subset J$, $k \geq 1$, such that $\bigcup_{x \in J_k} \left(x - \frac{1}{k}, x + \frac{1}{k}\right) \supset [-k, k]$. Let $J_k = \{x_i^k,\ i \in [p_k]\}$ be ordered in an increasing way. For simplicity, we also set $x_0^k := -\infty$ and $x_{p_k+1}^k := +\infty$. It is easily seen that for each $n \geq 0$ the sequence of σ-algebras

$$\mathcal{S}_n^k := \sigma^*\left(\left\{g_n^{-1}([x_{i-1}^k, x_i^k)),\ i \in [p_k + 1]\right\}\right), \quad k \geq 1,$$

increases to $\sigma^*(g_n)$. Moreover, for all $n \geq 0$ and $k \geq 1$

$$\mathbb{E}(h|\mathcal{S}_n^k) = \sum_{i=1}^{p_k+1} h_{I_{i,n}^k} \mathbb{I}_{I_{i,n}^k} \quad \text{a.e.},$$

where $I_{i,n}^k := g_n^{-1}([x_{i-1}^k, x_i^k))$, $h_{I_{i,n}^k} := \frac{1}{\mathrm{Leb}(I_{i,n}^k)} \int_{I_{i,n}^k} h(v)dv$ and $\mathbb{E}(\cdot|\cdot)$ denotes the conditional expectation on the probability space $([0,1], \mathcal{B}([0,1]), \mathrm{Leb})$. Thus, by Theorem 7.23 [24] and Remark A.1 (ii), for each $n \geq 0$

$$\mathbb{E}(h|\mathcal{S}_n^k) \to \mathrm{pr}_{g_n} h \quad \text{in} \quad L_2 \quad \text{as} \quad k \to \infty.$$

In particular, for every $n \geq 0$

$$\sup_{k \geq 1} \|\mathbb{E}(h|\mathcal{S}_n^k)\|_{L_2} = \|\mathrm{pr}_{g_n} h\|_{L_2}, \tag{A.6}$$

since $\mathcal{S}_n^k$, $k \geq 1$, increases and $\mathbb{E}(h|\mathcal{S}_n^k)$ is the projection of h in L_2 into the subspace of all $\mathcal{S}_n^k$-measurable functions.

Next, we fix $k \geq 1$ and $i \in [p_k + 1]$ such that $\mathrm{Leb}(I_{i,0}^k) > 0$ and denote the ends of $I_{i,0}^k$ by a and b, $a < b$. Then, using the monotonisity of the functions g_n, $n \geq 0$, the convergence of $\{g_n,\ n \geq 1\}$ to g_0 and the choice of J_k, we have that $a_n \to a$ and $b_n \to b$, where a_n and b_n are the ends of some intervals $I_{i_n,n}^k$. Consequently, for every $k \geq 1$

$$\mathbb{E}(h|\mathcal{S}_n^k) \to \mathbb{E}(h|\mathcal{S}_0^k) \quad \text{a.e.} \quad \text{as} \quad n \to \infty.$$

By Fatou's lemma, for every $k \geq 1$

$$\|\mathbb{E}(h|\mathcal{S}_0^k)\|_{L_2} \leq \varliminf_{n\to\infty} \|\mathbb{E}(h|\mathcal{S}_n^k)\|_{L_2}. \tag{A.7}$$

Hence,

$$\begin{aligned}\|\mathrm{pr}_{g_0} h\|_{L_2} &\overset{\text{(A.6)}}{=} \sup_{k\geq 1} \|\mathbb{E}(h|\mathcal{S}_0^k)\|_{L_2} \overset{\text{(A.7)}}{\leq} \sup_{k\geq 1} \varliminf_{n\to\infty} \|\mathbb{E}(h|\mathcal{S}_n^k)\|_{L_2} \\ &\overset{\text{(A.6)}}{\leq} \sup_{k\geq 1} \varliminf_{n\to\infty} \|\mathrm{pr}_{g_n} h\|_{L_2} = \varliminf_{n\to\infty} \|\mathrm{pr}_{g_n} h\|_{L_2}.\end{aligned}$$

The lemma is proved. ∎

A.3. Limit properties of some projection-valued functions. We recall that $\mathcal{L}_2(L_2)$ denotes the space of Hilbert Schmidt operators on L_2 with the inner product defined by (1.4) and the space $L_2([0,T], \mathcal{L}_2(L_2))$ is endowed with the inner product

$$(A,B)_{T,HS} = \int_0^T (A_t, B_t)_{HS} dt, \quad A, B \in L_2([0,T], \mathcal{L}_2(L_2)).$$

Since $\mathcal{L}_2(L_2)$ is a Hilbert space, $L_2([0,T], \mathcal{L}_2(L_2))$ also is a Hilbert space.

Proposition A.2. *Let functions* f *and* f^n, $n \geq 1$, *from* $C([0,T], L_2^{\uparrow})$ *satisfy the following conditions*

(a) $\{P^n,\ n \geq 1\}$ *converges weakly in* $L_2([0,T], \mathcal{L}_2(L_2))$ *to* P^∞, *that is,*

$$(P^n, A)_{T,HS} \to (P^\infty, A)_{T,HS} \quad \text{as } n \to \infty$$

for any $A \in L_2([0,T], \mathcal{L}_2(L_2))$, *where* $P_t^n = \mathrm{pr}_{f_t^n}$, $t \in [0,T]$;

(b) *there exists* $R \subseteq [0,T]$ *such that* $\mathrm{Leb}([0,T] \setminus R) = 0$ *and* $\|P_t h\|_{L_2} \leq \underline{\lim}_{n\to\infty} \|P_t^n h\|_{L_2}$ *for all* $t \in R$ *and* $h \in L_2$, *where* $P_t = \mathrm{pr}_{f_t}$, $t \in [0,T]$;

(c) *for every* $h \in L_2$ *and almost all* $t \in [0,T]$ $P_t^\infty(P_t h) = P_t^\infty h$.

Then $P^\infty = P$.

Remark A.3. (i) *Condition* (a) *together with the uniform boundedness principle imply the boundedness of the sequence* $\{P^n,\ n \geq 1\}$ *in* $L_2([0,T], \mathcal{L}_2(L_2))$.

(ii) *The function* P *belongs to* $L_2([0,T], \mathcal{L}_2(L_2))$ *and*

$$\|P\|_{T,HS} \leq \underline{\lim}_{n\to\infty} \|P^n\|_{T,HS},$$

by condition (b), *Fatou's lemma and the boundedness of* $\{P^n,\ n \geq 1\}$.

(iii) *Since* P_t^n *is an adjoint operator in* L_2 *for every* $t \in [0,T]$, P_t^∞ *is also adjoint for almost all* $t \in [0,T]$, *by Corollary A.1 below.*

To prove the proposition, we need to prove some auxiliary statements.

Lemma A.5. *Let* $\{e_i,\ i \in \mathbb{N}\}$ *be an orthonormal basis of* L_2 *and* $E_t^{i,j,r} = \mathbb{I}_{[0,r]}(t) e_i \otimes e_j$, $t \in [0,T]$, $i,j \in \mathbb{N}$, $r \in [0,T]$. *Then* $\mathrm{span}\{E^{i,j,r},\ r \in [0,T],\ i,j \in \mathbb{N}\}$ *is dense in* $L_2([0,T], \mathcal{L}_2(L_2))$.

Proof. The statement easily follows from the density of simple functions

$$\sum_{k=1}^{n} \mathbb{I}_{[t_{k-1}, t_k)} A_k$$

in $L_2([0,T], \mathcal{L}_2(L_2))$, where $0 = t_0 < t_1 < \ldots < t_n = T$ and $A_k \in \mathcal{L}_2(L_2)$, $k \in [n]$, and the fact that $\{e_i \otimes e_j,\ i,j \in \mathbb{N}\}$ is an orthonormal basis of $\mathcal{L}_2(L_2)$. □

Corollary A.1. *Let* $\{e_i,\ i \in \mathbb{N}\}$ *be an orthonormal basis of* L_2 *and* $A, B \in L_2([0,T], \mathcal{L}_2(L_2))$. *If for each* $r \in [0,T]$ *and* $i,j \in \mathbb{N}$

$$\int_0^r (A_t e_i, e_j)_{L_2} dt = \int_0^r (B_t e_i, e_j)_{L_2} dt,$$

then $A = B$.

PROOF. The statement immediately follows from Lemma A.5 and the equality

$$(A, E^{i,j,r})_{T,HS} = \int_0^r (A_t e_i, e_j)_{L_2} dt.$$

$\square$

PROOF OF PROPOSITION A.2. Let $e \in L_2([0,T], L_2)$ such that

$$\|e_t\|_{L_2} = 1 \quad \text{and} \quad P_t e_t = e_t \quad \text{for almost all } t \in [0,T]. \tag{A.8}$$

We first prove that

$$(P_t^\infty e_t, e_t) = 1 \quad \text{for almost all } t \in [0,T]. \tag{A.9}$$

To show this, we set for fixed $r \in [0,T]$

$$A_t^r := \mathbb{I}_{[0,r]}(t) e_t \otimes e_t, \quad t \in [0,T],$$

and use the weak convergence of P^n to P^∞. So,

$$\begin{aligned} r = \int_0^r \|e_t\|_{L_2}^2 dt = \int_0^r \|P_t e_t\|_{L_2}^2 dt &\overset{(b)}{\leq} \int_0^r \varliminf_{n\to\infty} \|P_t^n e_t\|_{L_2}^2 dt \\ &\overset{\text{Fatou's lemma}}{\leq} \varliminf_{n\to\infty} \int_0^r \|P_t^n e_t\|_{L_2}^2 dt = \varliminf_{n\to\infty} \int_0^r (P_t^n e_t, e_t)_{L_2} dt \\ &= \varliminf_{n\to\infty} (A^r, P^n)_{T,HS} \overset{(a)}{=} (A^r, P^\infty)_{T,HS} = \int_0^r (P_t^\infty e_t, e_t) dt. \end{aligned}$$

On the other hand, $(P_t^n e_t, e_t)_{L_2} = \|P_t^n e_t\|_{L_2}^2 \leq \|e_t\|_{L_2}^2 = 1$ for all $t \in [0,T]$ and $n \geq 1$. Hence,

$$\int_0^r (P_t^\infty e_t, e_t) dt = \lim_{n\to\infty} \int_0^r (P_t^n e_t, e_t) dt \leq r.$$

Consequently,

$$\int_0^r (P_t^\infty e_t, e_t) dt = r$$

for all $r \in [0,T]$. This immediately implies (A.9).

Next, without loss of generality, we may suppose that $f_t \in D^\uparrow$ for all $t \in [0,T]$. We set for each $v \in (0,1)$

$$e_t^v(u) = \frac{1}{\sqrt{m_{f_t}(v)}} \mathbb{I}_{\{f_t(v) = f_t(u)\}}, \quad u \in (0,1), \quad t \in [0,T],$$

where m_{f_t} is defined by (A.4). By Remark A.3 (ii), $\int_0^T \|P_t\|_{HS}^2 dt < \infty$. Thus, $f_t \in \mathcal{S}^\uparrow$ for almost all $t \in [0,T]$, by Lemma A.3. This together with the right continuity of $f_t(u),\ u \in (0,1)$, imply that for every $v \in (0,1)$ the function e_t^v is well-defined for almost all t and $e^v \in L_2([0,T], L_2)$. Let

$$e_t^{v_1,v_2} := \begin{cases} 1, & f_t(v_1) = f_t(v_2), \\ \frac{e_t^{v_1}+e_t^{v_2}}{\sqrt{2}}, & f_t(v_1) \neq f_t(v_2), \end{cases} \quad t \in [0,T].$$

It is easy to see that e^{v_1,v_2} belong to $L_2([0,T], L_2)$ for all $v_1, v_2 \in (0,1)$.

Since e^{v_1,v_2} and e^{v_1} satisfy (A.8) for all $v_1, v_2 \in (0,1)$,

(A.10)
$$(P_t^\infty e_t^{v_1,v_2}, e_t^{v_1,v_2}) = 1 \quad \text{and} \quad (P_t^\infty e_t^{v_1}, e_t^{v_1}) = 1 \quad \text{for almost all } t \in [0,T].$$

We set

$$\begin{aligned} R = \Big\{ t \in [0,T] :\ &(P_t^\infty e_t^{v_1,v_2}, e_t^{v_1,v_2}) = 1 \\ &\text{and } (P_t^\infty e_t^{v_1}, e_t^{v_1}) = 1,\ v_1, v_2 \in (0,1) \cap \mathbb{Q} \Big\} \\ \cap \Big\{ t \in [0,T] :\ &P_t^\infty (P_t) = P_t^\infty \\ &\text{and } \|P_t\|_{HS} < \infty \} \cap \{ t \in [0,T] :\ P_t^\infty \text{ is adjoint} \Big\}. \end{aligned}$$

Then $\mathrm{Leb}([0,T] \setminus R) = 0$, by (A.10), Condition (c) and Remark A.3 (ii), (iii).

Next, we fix $t \in R$ and note that f_t is a step function with a finite number of values, by Lemma A.3. Thus, there exists $v_1, \dots, v_l$ from $(0,1) \cap \mathbb{Q}$, which depends on t, such that $l = \# f_t$ and $\{e_i := e_t^{v_i},\ i = 1, \dots, l\}$ is an orthonormal basis of the image of P_t. We extend $\{e_i,\ i = 1, \dots, l\}$ to an orthonormal basis of L_2 denoted by $\{e_i,\ i \in \mathbb{N}\}$ and note that $f_t(v_i) \neq f_t(v_j)$ for $i \neq j$, according to the definition of e^v. By the choice of t, $(P_t^\infty e_i, e_i) = 1$, $i = 1, \dots, l$. Moreover, $(P_t^\infty e_i, e_j) = 0$ for all $i, j \in [l]$ and $i \neq j$. Indeed,

$$\begin{aligned} 1 = (P_t^\infty e_t^{v_i,v_j}, e_t^{v_i,v_j}) &= \frac{1}{2}(P_t^\infty (e_i + e_j), e_i + e_j) \\ &= \frac{1}{2}\big[(P_t^\infty e_i, e_i) + (P_t^\infty e_j, e_j) + 2(P_t^\infty e_i, e_j)\big] \\ &= 1 + (P_t^\infty e_i, e_j). \end{aligned}$$

If $i > l$, then

$$P_t^\infty e_i = P_t^\infty (P_t e_i) = P_t^\infty 0 = 0,$$

by Condition (c). This implies that $(P_t^\infty e_i, e_j) = (P_t e_i, e_j)$ for all $i, j \in \mathbb{N}$. Thus, $P_t = P_t^\infty$. The proposition is proved. $\square$

A.4. Quadratic variations of L2-valued continuous semimartingales. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(\mathcal{F}_t)_{t\in[0,T]}$ be a complete right continuous filtration.

PROPOSITION A.3. *Let $g \in L_2^{\uparrow}$, M_t, $t \in [0,T]$, be a continuous L_2-valued square integrable $(\mathcal{F}_t)$-martingale with the quadratic variation*

$$\langle M\rangle_t = \int_0^t L_s L_s^* ds,$$

where L_t, $t \in [0,T]$, is an $(\mathcal{F}_t)$-adapted $\mathcal{L}_2(L_2)$-valued process belonging to $L_2([0,T], \mathcal{L}_2(L_2))$ a.s. and L_s^* denotes the adjoint operator of L_s. Let b_t, $t \in [0,T]$, be an $(\mathcal{F}_t)$-adapted L_2-valued continuous process such that for each $h \in L_2$ the process $(b_t, h)_{L_2}$, $t \in [0,T]$, has a locally finite variation. Let also the process

$$X_t := g + M_t + b_t, \quad t \in [0,T],$$

take values in $L_2^{\uparrow}$. Then

$$\mathbb{P}\left\{ \exists R \subseteq [0,T] \ s.t. \quad \begin{array}{l} \mathrm{Leb}\{[0,T]\setminus R\} = 0 \ \ and \\ L_t(\mathrm{pr}_{X_t} h) = L_t h, \ \forall t \in R, \ \forall h \in L_2 \end{array} \right\} = 1.$$

To prove the proposition, we need the following lemma.

LEMMA A.6. *Let $x(t)$, $t \in [0,T]$, be a continuous real valued semimartingale. Then*

$$\int_0^T \mathbb{I}_{\{0\}}(x(t)) d[x]_t = 0 \quad a.s.$$

PROOF. The statement immediately follows from the equality

$$\int_0^T \mathbb{I}_{\{0\}}(x(t)) d[x]_t = \int_{-\infty}^{+\infty} \mathbb{I}_{\{0\}}(y) l_T^y dy = 0,$$

where l_t^y, $t \in [0,T]$, $y \in \mathbb{R}$, is the local time of x (see e.g. Theorem 22.5 [24]). $\square$

PROOF OF PROPOSITION A.3. We set

$$f_{a,b} := \frac{1}{b-a}\mathbb{I}_{[a,b)} \tag{A.11}$$

for each $a, b \in [0,1]$, $a < b$, and

$$\mathcal{R} := \{f_{a,b} : \ a, b \in [0,1] \cap \mathbb{Q}, \ a < b\}.$$

If $b_1 \leq a_2$, then we will write $f_{a_1,b_1} \preccurlyeq f_{a_2,b_2}$.

Taking $f', f'' \in \mathcal{R}$, $f' \preccurlyeq f''$ and applying Lemma A.6 to the semimartingale

$$x(t) := X_t(f'') - X_t(f') = X_t(f'' - f'), \quad t \in [0,T],$$

where $X_t(f) := (X_t, f)_{L_2}$, we obtain

$$\begin{aligned} 0 &= \int_0^T \mathbb{I}_{\{0\}}(X_t(f'') - X_t(f'))d[X(f'' - f'')]_t \\ &= \int_0^T \mathbb{I}_{\{0\}}(X_t(f'') - X_t(f'))\|L_t(f'' - f')\|_{L_2}^2 dt \quad \text{a.s.} \end{aligned}$$

For each $\omega \in \Omega$, we set

$$\begin{aligned} R(\omega) := \big\{ t \in [0,T] : \mathbb{I}_{\{0\}}(X_t(f'')(\omega) - X_t(f')(\omega))\|L_t(\omega)(f'' - f')\|_{L_2} = 0, \\ \forall f', f'' \in \mathcal{R},\ f' \preccurlyeq f'' \big\} \end{aligned}$$

and

$$\Omega' = \{\omega : \ \mathrm{Leb}([0,T] \setminus R(\omega)) = 0\}.$$

Since $\mathcal{R}$ is countable, we have that $\mathbb{P}\{\Omega'\} = 1$. Next, let $\omega \in \Omega'$ and $t \in R(\omega)$ be fixed. To finish the proof of the theorem, it is needed to show that

$$L_t(\omega)(\mathrm{pr}_{X_t(\omega)} h) = L_t(\omega)h \tag{A.12}$$

for all $h \in L_2$. But since $C([0,1])$ is dense in L_2, the equality is enough to check only for $h \in C([0,1])$. So, we fix $h \in C([0,1])$ and denote the modification of $X_t(\omega)$ from $D^{\uparrow}$ also by $X_t(\omega)$.

First, we take arbitrary $a < b$ from $(0,1) \cap \mathbb{Q}$ such that $X_t(a,\omega) = X_t(b,\omega)$ and show that

$$L_t(\omega)h = L_t(\omega)(h\mathbb{I}_{\pi^c} + h_\pi \mathbb{I}_\pi), \tag{A.13}$$

where $h_\pi := \frac{1}{b-a}\int_a^b h(u)du$, $\pi := [a,b]$ and $\pi^c := [0,1] \setminus \pi$. Let $a = u_0 < u_1 < \ldots < u_k = b$ be an arbitrary partition of $[a,b]$ with $u_i \in \mathbb{Q}$, $i \in [k]$. The monotonisity of $X_t(u,\omega)$, $u \in (0,1)$, yields that $X_t(f_{u_{i-1},u_i})(\omega) = X_t(f_{u_{j-1},u_j})(\omega)$ for all $i,j \in [k]$. So, we have that

$$L_t(\omega)f_{u_{i-1},u_i} = L_t(\omega)f_{u_{j-1},u_j},\ i,j \in [k],$$

due to the choice of t and ω, where f_{u_{i-1},u_i}, f_{u_{j-1},u_j} are defined by (A.11). Using the equality $f_{a,b} = \sum_{i=1}^k \frac{u_i - u_{i-1}}{b-a} f_{u_{i-1},u_i}$, one can easily seen that

$$L_t(\omega)f_{u_{i-1},u_i} = L_t(\omega)f_{a,b} \tag{A.14}$$

for all $i \in [k]$. Taking

$$h^k := h\mathbb{I}_{\pi^c} + \sum_{i=1}^{k} h(u_{i-1})\mathbb{I}_{[u_{i-1},u_i)}$$

and using (A.14), we obtain

$$\begin{aligned} L_t(\omega)h^k &= L_t(\omega)(h\mathbb{I}_{\pi^c}) + \sum_{i=1}^{k} h(u_{i-1})L_t(\omega)\mathbb{I}_{[u_{i-1},u_i)} \\ &= L_t(\omega)(h\mathbb{I}_{\pi^c}) + \sum_{i=1}^{k} h(u_{i-1})(u_i - u_{i-1})L_t(\omega)f_{u_{i-1},u_i} \\ &= L_t(\omega)(h\mathbb{I}_{\pi^c}) + \sum_{i=1}^{k} h(u_{i-1})(u_i - u_{i-1})L_t(\omega)f_{a,b} = L_t(\omega)\tilde{h}^k, \end{aligned}$$

where

$$\tilde{h}^k = h\mathbb{I}_{\pi^c} + \frac{1}{b-a}\sum_{i=1}^{k} h(u_{i-1})(u_i - u_{i-1})\mathbb{I}_{\pi}.$$

Since $h^k \to h$ and $\tilde{h}^k \to h\mathbb{I}_{\pi^c} + h_\pi \mathbb{I}_\pi$ in L_2 as $\max_{i\in[k]}(u_i - u_{i-1}) \to 0$, the equality (A.13) holds.

Next, by the approximation argument, it is easy to prove that (A.13) folds for each $\pi \in \mathcal{K}_{X_t(\omega)}$, where $\mathcal{K}_g$ was defined in Section A.2 for any $g \in D^{\uparrow}$. Let $\mathcal{K}_{X_t(\omega)} = \{\pi_i,\ i \in \mathbb{N}\}$, that is countable, be ordered in decreasing of the length of π_i. If $\mathcal{K}_g$ is finite then (A.12) immediately follows from (A.13) and Lemma A.1. Otherwise, using (A.13), the continuity of $L_t(\omega)$ and Lemma A.1, we have

$$\begin{aligned} L_t(\omega)h &= L_t(\omega)\left(h\mathbb{I}_{\tilde{\pi}_l} + \sum_{i=1}^{l} h_{\pi_i}\mathbb{I}_{\pi_i}\right) \\ &\quad \to L_t(\omega)\left(h_{X_t(\omega)}\right) = L_t(\omega)\left(\mathrm{pr}_{X_t(\omega)}\, h\right) \quad \text{as} \ \ l \to \infty, \end{aligned}$$

where h_g is defined by (A.3) and $\tilde{\pi}_l := [0,1] \setminus \left(\bigcup_{i=1}^{l} \pi_i\right)$. This finishes the proof of the proposition. $\square$

A.5. Some compact sets in Skorohod space. Let (E, r) be a Polish space and let $D([a,b], E)$ denote the space of càdlág functions from $[a,b]$ to E which are continuous at b. We endow $D([a,b], E)$ with the metric

$$d_{[a,b]}(f,g) = \inf_{\lambda \in \Lambda_{[a,b]}} \left\{\gamma(\lambda) \vee \sup_{u\in[a,b]} r(f(\lambda(u)), g(u))\right\}, \quad f, g \in D([a,b], E),$$

where $\Lambda_{[a,b]}$ is the set of all strictly increasing functions $\lambda : [a,b] \to [a,b]$ such that $\lambda(a) = a$, $\lambda(b) = b$ and

$$\gamma(\lambda) := \sup_{v<u} \left| \log \frac{\lambda(u) - \lambda(v)}{u - v} \right| < \infty.$$

For each $[c,d] \subset [a,b]$ and $f \in D([a,b],E)$ it is clear that the function

$$f^{[c,d]}(u) := \begin{cases} f(u), & u \in [c,d), \\ f(d-), & u = d. \end{cases}$$

belongs to $D([c,d],E)$.

Proposition A.4. *Let $U = \{u_i,\ i = 0,\dots,l\}$ be an ordered partition of $[a,b]$ and let $\{X_n,\ n \geq 1\}$ be an arbitrary sequence of random elements in $D([a,b],E)$. If $\left\{X_n^{[u_{i-1},u_i]},\ n \geq 1\right\}$ is tight in $D([u_{i-1},u_i],E)$ for any $i \in [l]$, then $\{X_n,\ n \geq 1\}$ is tight in $D([a,b],E)$.*

Proof. Let K_i be compact in $D([u_{i-1},u_i],E)$, $i \in [l]$, and let

$$K := \left\{ f \in D([a,b],E) :\ f^{[u_{i-1},u_i]} \in K_i,\ i \in [l] \right\}. \tag{A.15}$$

In order to prove the proposition, it is enough to show that K is compact in $D([a,b],E)$. Indeed, by the definition of the tightness (see e.g. Section 3.2 [16]), for each $\varepsilon > 0$ there exist compact sets K_i, $i \in [l]$, such that

$$\mathbb{P}\left\{ X_n^{[u_{i-1},u_i]} \notin K_i \right\} \leq \frac{\varepsilon}{l}$$

for all $i \in [l]$ and $n \geq 1$. Thus,

$$\mathbb{P}\{X_n \notin K\} = \mathbb{P}\left\{ \bigcup_{i=1}^{l} \left\{ X_n^{[u_{i-1},u_i]} \notin K_i \right\} \right\} \leq \varepsilon,$$

where K is defined by (A.15). This implies the tightness of $\{X_n,\ n \geq 1\}$ in $D([a,b],E)$.

So, let $\{f_n,\ n \geq 1\} \subset K$. Then there exists a subsequence $N \subset \mathbb{N}$ such that $f_n^{[u_{i-1},u_i]}$ converges to f^i in $D([u_{i-1},u_i],E)$ along N for any $i \in [l]$. Thus, for every $i \in [l]$ there exists a sequence $\{\lambda_n^i,\ n \in N\} \subset \Lambda_{[u_{i-1},u_i]}$ such that

$$\gamma(\lambda_n^i) \to 0 \quad \text{and} \quad \sup_{u \in [u_{i-1},u_i]} r\left(f_n^{[u_{i-1},u_i]}(\lambda_n^i(u)), f^i(u) \right) \to 0 \quad \text{along} \quad N.$$

Taking

$$f(u) := \sum_{i=1}^{l} f^i(u)\mathbb{I}_{[u_{i-1},u_i)}(u) + f^l(u)\mathbb{I}_{\{u_l\}}(u)$$

and

$$\lambda_n := \sum_{i=1}^{l} \lambda_n^i(u)\mathbb{I}_{[u_{i-1},u_i)}(u) + \lambda_n^l(u)\mathbb{I}_{\{u_l\}}(u), \quad n \geq 1,$$

it is easily seen that $f \in D([a,b],E)$ and λ_n is a continuous strictly increasing function from $[a,b]$ onto $[a,b]$ for all $n \geq 1$. Moreover,

$$\sup_{u\in[a,b]} |\lambda_n(u) - u| \to 0 \quad \text{and} \quad \sup_{u\in[a,b]} r(f_n(\lambda_n(u)), f(u)) \to 0 \quad \text{along} \quad N.$$

By Theorem 12.1 [5], f_n converges to f in $D([a,b],E)$ along N. The proposition is proved. □

REFERENCES

[1] Alonso, A. and Brambila-Paz, F. (1998). L^p-continuity of conditional expectations. *J. Math. Anal. Appl.* **221** 161–176. MR1619139

[2] Archer, A. J. and Rauscher, M. (2004). Dynamical density functional theory for interacting Brownian particles: stochastic or deterministic? *J. Phys. A* **37** 9325–9333. MR2095422

[3] Arratia, R. A. (1979). *Coalescing Brownian motion on the line*. ProQuest LLC, Ann Arbor, MI Thesis (Ph.D.)–The University of Wisconsin - Madison. MR2630231

[4] Berestycki, N., Garban, C. and Sen, A. (2015). Coalescing Brownian flows: a new approach. *Ann. Probab.* **43** 3177–3215. MR3433579

[5] Billingsley, P. (1999). *Convergence of probability measures*, second ed. *Wiley Series in Probability and Statistics: Probability and Statistics*. John Wiley & Sons, Inc., New York A Wiley-Interscience Publication. MR1700749

[6] Cherny, A. S. and Engelbert, H.-J. (2005). *Singular stochastic differential equations*. *Lecture Notes in Mathematics* **1858**. Springer-Verlag, Berlin. MR2112227

[7] Dawson, D. A. (1993). Measure-valued Markov processes. In *École d'Été de Probabilités de Saint-Flour XXI—1991*. *Lecture Notes in Math.* **1541** 1–260. Springer, Berlin. MR1242575

[8] de la Torre, J. A., Espanol, P. and Donev, A. (2015). Finite element discretization of non-linear diffusion equations with thermal fluctuations. *The Journal of Chemical Physics* **142** 094115.

[9] Dean, D. S. (1996). Langevin equation for the density of a system of interacting Langevin processes. *J. Phys. A* **29** L613–L617. MR1446882

[10] Delfau, J.-B., Ollivier, H., López, C., Blasius, B. and Hernández-Garcí a, E. (2016). Pattern formation with repulsive soft-core interactions: discrete particle dynamics and Dean-Kawasaki equation. *Phys. Rev. E* **94** 042120, 13. MR3744636

[11] Donev, A., Fai, T. G. and Vanden-Eijnden, E. (2014). A reversible mesoscopic model of diffusion in liquids: from giant fluctuations to Fick's law. *Journal of Statistical Mechanics: Theory and Experiment* **2014** P04004.

[12] Donev, A. and Vanden-Eijnden, E. (2014). Dynamic density functional theory with hydrodynamic interactions and fluctuations. *The Journal of Chemical Physics* **140** 234115.

[13] Dorogovtsev, A. A. (2004). One Brownian stochastic flow. *Theory Stoch. Process.* **10** 21–25. MR2329772

[14] Dorogovtsev, A. A. and Ostapenko, O. V. (2010). Large deviations for flows of interacting Brownian motions. *Stoch. Dyn.* **10** 315–339. MR2671379

[15] Engelbert, H.-J. and Peskir, G. (2014). Stochastic differential equations for sticky Brownian motion. *Stochastics* **86** 993–1021. MR3271518

[16] Ethier, S. N. and Kurtz, T. G. (1986). *Markov processes. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics*. John Wiley & Sons, Inc., New York Characterization and convergence. MR838085

[17] Fontes, L. R. G., Isopi, M., Newman, C. M. and Ravishankar, K. (2004). The Brownian web: characterization and convergence. *Ann. Probab.* **32** 2857–2883. MR2094432

[18] Frusawa, H. and Hayakawa, R. (2000). On the controversy over the stochastic density functional equations. *Journal of Physics A: Mathematical and General* **33** L155.

[19] Gawarecki, L. and Mandrekar, V. (2011). *Stochastic differential equations in infinite dimensions with applications to stochastic partial differential equations. Probability and its Applications (New York)*. Springer, Heidelberg. MR2560625

[20] Howitt, C. and Warren, J. (2009). Consistent families of Brownian motions and stochastic flows of kernels. *Ann. Probab.* **37** 1237–1272. MR2546745

[21] Ikeda, N. and Watanabe, S. (1981). *Stochastic differential equations and diffusion processes. North-Holland Mathematical Library* **24**. North-Holland Publishing Co., Amsterdam-New York; Kodansha, Ltd., Tokyo. MR637061

[22] Jacod, J. and Shiryaev, A. N. (2003). *Limit theorems for stochastic processes*, second ed. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **288**. Springer-Verlag, Berlin. MR1943877

[23] Jakubowski, A. (1986). On the Skorokhod topology. *Ann. Inst. H. Poincaré Probab. Statist.* **22** 263–285. MR871083

[24] Kallenberg, O. (2002). *Foundations of modern probability*, second ed. *Probability and its Applications (New York)*. Springer-Verlag, New York. MR1876169

[25] Kawasaki, K. (1994). Stochastic model of slow dynamics in supercooled liquids and dense colloidal suspensions. *Physica A: Statistical Mechanics and its Applications* **208** 35 - 64.

[26] Kim, B., Kawasaki, K., Jacquin, H. and van Wijland, F. (2014). Equilibrium dynamics of the Dean-Kawasaki equation: Mode-coupling theory and its extension. *Phys. Rev. E* **89** 012150.

[27] Konarovskyi, V. (2017a). A system of coalescing heavy diffusion particles on the real line. *Ann. Probab.* **45** 3293–3335.

[28] Konarovskyi, V. (2017b). On asymptotic behavior of the modified Arratia flow. *Electron. J. Probab.* **22** 31.

[29] Konarovskyi, V., Lehmann, T. and von Renesse, M.-K. (2019). Dean-Kawasaki dynamics: ill-posedness vs. triviality. *Electron. Commun. Probab.* **24** Paper No. 8, 9. MR3916340

[30] Konarovskyi, V., Lehmann, T. and von Renesse, M.-K. On Dean-Kawasaki Dynamics with smooth Drift Potential. *arXiv:1812.11068*.

[31] Konarovskyi, V. and von Renesse, M. Reversible Coalescing-Fragmentating Wasserstein Dynamics on the Real Line. *arXiv:1709.02839*.

[32] Konarovskyi, V. and von Renesse, M. A diffuse Ornstein-Uhlenbeck process. *in preparation.*

[33] Konarovskyi, V. and von Renesse, M.-K. (2019). Modified massive Arratia flow and Wasserstein diffusion. *Comm. Pure Appl. Math.* **72** 764–800. MR3914882

[34] Le Jan, Y. and Raimond, O. (2004). Flows, coalescence and noise. *Ann. Probab.* **32** 1247–1315. MR2060298

[35] Ma, Z. M. and Röckner, M. (1992). *Introduction to the theory of (nonsymmetric) Dirichlet forms. Universitext.* Springer-Verlag, Berlin. MR1214375

[36] Marconi, U. M. B. and Tarazona, P. (1999). Dynamic density functional theory of fluids. *The Journal of Chemical Physics* **110** 8032-8044.

[37] Marconi, U. M. B. and Tarazona, P. (2000). Dynamic density functional theory of fluids. *Journal of Physics: Condensed Matter* **12** A413.

[38] Marx, V. (2018). A new approach for the construction of a Wasserstein diffusion. *Electron. J. Probab.* **23** 54 pp.

[39] Munakata, T. (1977). Liquid instability and freezing—reductive perturbation approach. *J. Phys. Soc. Japan* **43** 1723–1728. MR0462381

[40] Munakata, T. (1990). Density Fluctuations in Liquids -Application of a Dynamical Density Functional Theory-*. *Journal of the Physical Society of Japan* **59** 1299-1304.

[41] Revuz, D. and Yor, M. (1999). *Continuous martingales and Brownian motion*, third ed. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **293**. Springer-Verlag, Berlin. MR1725357

[42] Riabov, G. V. (2017). Random dynamical systems generated by coalescing stochastic flows on $\mathbb{R}$. *Stoch. Dyn.* **18** 1850031, 24.

[43] Schertzer, E., Sun, R. and Swart, J. M. (2014). Stochastic flows in the Brownian web and net. *Mem. Amer. Math. Soc.* **227** vi+160. MR3155782

Universität Leipzig
Fakultät für Mathematik und Informatik
Augustusplatz 10
04109 Leipzig, Germany
E-mail: konarovskyi@gmail.com